

CANONICAL CONNECTION AND CONTACT CAUCHY-RIEMANN MAPS ON CONTACT MANIFOLDS I

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ABSTRACT. In the present article, we develop the analysis of the following nonlinear elliptic system of equations

$$\bar{\partial}^\pi w = 0, \quad d(w^* \lambda \circ j) = 0$$

associated to each given contact triad (Q, λ, J) on a contact manifold (Q, ξ) , which was first introduced by Hofer [H2]. *We directly work with this equation on the contact manifolds without involving the symplectization process.*

We first establish local a priori $W^{k,2}$ coercive estimates for all $k \geq 2$ in terms of $\|dw\|_{L^2}$, $\|dw\|_{L^4}$. Equipping the punctured Riemann surface $(\tilde{\Sigma}, j)$ with a Kähler metric h on $\tilde{\Sigma}$ that is cylindrical on a puncture neighborhood $D \setminus \{r\}$ at each marked point $r \in \Sigma$ and the associated isothermal coordinates (τ, t) , we prove the asymptotic (subsequence) convergence to the ‘spiraling’ instantons along the ‘rotating’ Reeb orbit for any solution w , not necessarily of $w^* \lambda \circ j$ being exact (i.e., allowing non-zero ‘charge’ $a \neq 0$), with bounded gradient $\|\nabla w\|_{C^0} < C$ and finite π -harmonic energy

$$E_{(\lambda, J; D \setminus \{0\})}(w, j) = \frac{1}{2} \int_{D \setminus \{r\}} |d^\pi w|^2 < \infty$$

on $D \setminus \{r\}$. We also prove the exponential convergence to a Reeb orbit when $a = 0$ and the Reeb orbit is nondegenerate. The case with non-vanishing charge and the Morse-Bott case will be treated in a sequel [OW2]. We do pedestrian tensorial calculations in our estimates using the contact triad connection ∇ on Q and the contact Hermitian connection on the Hermitian vector bundle $(\xi, d\lambda|_\xi, J)$ introduced in [OW1]. Our tensor calculations also clarify the geometry behind Hofer-Wysocki-Zehnder’s coordinate calculations involved in their study of exponential decay estimates even for pseudoholomorphic curves on the symplectization of contact manifolds (or on symplectic manifolds with cylindrical ends).

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CONTENTS

1. Introduction	2
2. Contact triad connection and contact Hermitian connection	8
Part 1. A priori estimates and asymptotic convergence	9
3. Linearization operator of Reeb orbits	10
4. Tensorial calculations for geometric energy density function	13
5. $W^{k,2}$ -coercive estimates: the closed case	18
5.1. $W^{2,2}$ estimates	18
5.2. $W^{k,2}$ estimates for $k \geq 3$	20
6. Asymptotic convergence to Reeb orbits at puncture	23
7. Fundamental equation in cylindrical coordinates	26
8. $W^{k,2}$ -coercive estimates: the punctured case	28
9. Appendix: Wedge product of vector-valued forms	31
Part 2. C^∞ exponential estimates on cylindrical ends	33
10. Asymptotic perturbation results of eigenvalues	34
11. L^2 exponential convergence of ξ -component of dw	38
12. Alternating bootstrapping and C^∞ exponential convergence of dw	45
13. C^0 exponential convergence of w (and of a)	49
13.1. C^0 exponential convergence of the map w	50
13.2. C^0 exponential convergence of a in the symplectization case	51
References	52

1. INTRODUCTION

A contact manifold (Q, ξ) is a $2n+1$ dimensional manifold equipped with a completely non-integrable distribution of rank $2n$, called a contact structure. Complete non-integrability of ξ can be expressed by the non-vanishing property

$$\lambda \wedge (d\lambda)^n \neq 0$$

for a one-form λ which defines the distribution, i.e., $\ker \lambda = \xi$. Such a one-form λ is called a contact form associated to ξ . Associated to the given contact form λ , we have the unique vector field X_λ determined by

$$X_\lambda \lrcorner \lambda \equiv 1, \quad X_\lambda \lrcorner d\lambda \equiv 0.$$

In relation to the study of pseudo-holomorphic curves, one consider an endomorphism $J : \xi \rightarrow \xi$ with $J^2 = -id$ and regard (ξ, J) as a complex vector bundle. In the presence of the contact form λ , one usually considers the set of J that is compatible to $d\lambda$ in the sense that the bilinear form $g_\xi = d\lambda(\cdot, J\cdot)$ defines a Hermitian vector bundle (ξ, J, g_ξ) on Q . We call a triple (Q, λ, J) a *contact triad*.

Motivated by the already well-established Gromov's theory of pseudoholomorphic curves, in his study of Weinstein's conjecture in 3 dimension for over-twisted contact structure, Hofer [H1] (and others follow) studied pseudoholomorphic curves in the symplectization $W = \mathbb{R}_+ \times Q$ with symplectic form $\omega = d(r\lambda)$ or in $\mathbb{R} \times Q$

with $\omega = d(e^s \lambda)$ with $r = e^s$ with cylindrical almost complex structure given by

$$J = J_0 \oplus J, \quad TW \cong \mathbb{R} \left\{ \frac{\partial}{\partial s} \right\} \oplus \mathbb{R} \{X_\lambda\} \oplus \xi.$$

A map $w : (\Sigma, j) \rightarrow W$ is J -holomorphic if and only if its components $a = s \circ u$ and $w = \pi \circ u$ satisfy the equation

$$\begin{cases} \pi_\lambda \left(\frac{\partial w}{\partial \tau} \right) + J(w) \pi_\lambda \left(\frac{\partial w}{\partial t} \right) = 0 \\ w^* \lambda \circ j = da. \end{cases} \quad (1.1)$$

Because of the factor \mathbb{R} which is noncompact, the relevant analysis becomes somewhat subtle to deal with the noncompactness of symplectization. (See [HWZ1, HWZ2] for the relevant elliptic estimates in the symplectization of the triad (Q, λ, J) in 3-dimension.)

Hofer-Wysocki-Zehnder [HWZ1, HWZ2] derived exponential decay estimates of proper pseudoholomorphic curves in symplectization by brute force coordinate calculations which largely rely on a choice of some special coordinates around the given Reeb orbit and involve quite complex coordinate calculations. Bourgeois [Bo, BEHWZ] amplified similar coordinate calculations for the case of Morse-Bott with even bigger complexity. In addition, their higher order exponential estimates is somewhat unsatisfying in that *they use C^∞ decay (or rather C^2 decay) from the beginning to prove the C^1 exponential convergence. As a result, their C^∞ -exponential convergence proof does not entirely follow the standard bootstrap argument type.* There also appeared several zero-order operators in their derivations whose geometric meanings were not clear to the authors. Our attempt to improve the presentation of this decay estimate and to understand the geometric meaning of these operators in the tensorial language was the starting point of the research performed in the present paper and [OW1], [OW2].

For each given triad structure (Q, λ, J) , we equip Q with the triad metric

$$g = d\lambda(\cdot, J\cdot) + \lambda \otimes \lambda$$

and decompose $d^\pi w := \pi dw = \partial^\pi w + \bar{\partial}^\pi w$ for the map $w : \Sigma \rightarrow Q$ with J -linear and anti- J -linear part of the derivative dw as a $w^* \xi$ -valued one-forms on Σ .

We start with the maps w satisfying just $\bar{\partial}^\pi w = 0$, which is a nonlinear degenerate elliptic equation.

Definition 1.1 (Contact Cauchy-Riemann map). Let (Q, λ, J) be a contact triad and let (Σ, j) be a Riemann surface. We call any map $w : \Sigma \rightarrow Q$ a *contact Cauchy-Riemann map* if it satisfies $\bar{\partial}^\pi w = 0$.

In [OW1], we introduced a canonical connection, called the contact triad connection, on each contact triad (Q, λ, J) . Using the tensor calculus based on this connection, we establish the analysis of the contact Cauchy-Riemann maps in the present paper and in a sequel [OW2].

We denote by ∇ the contact triad connection on Q and ∇^π the contact Hermitian connection on the Hermitian vector bundle $(\xi, d\lambda|_\xi, J)$ introduced in [OW1]. Various symmetry properties carried by the connections ∇ and ∇^π enable us to derive the following precise formulae concerning the second covariant differential of w and the Laplacian of the π -harmonic energy density function for any contact Cauchy-Riemann map w .

Theorem 1.2 (Fundamental equation). *Let w be a contact Cauchy-Riemann map. Then*

$$\begin{aligned} d^{\nabla^\pi}(d^\pi w) &= d^{\nabla^\pi}(\partial^\pi w) \\ &= -w^* \lambda \circ j \wedge \left(\frac{1}{2} (\mathcal{L}_{X_\lambda} J) \partial^\pi w \right). \end{aligned}$$

We define the ξ -component of the standard harmonic energy density function by

$$e^\pi := |d^\pi w|^2 = |\pi dw|^2.$$

Definition 1.3. For any smooth map $w : \dot{\Sigma} \rightarrow Q$,

$$E_{(\lambda, J)}(w, j) = \frac{1}{2} \int_{\dot{\Sigma}} |d^\pi w|^2$$

and call it the π -harmonic energy of a smooth map.

Theorem 1.4. *Let w be a contact Cauchy-Riemann map. Then*

$$\begin{aligned} \frac{1}{2} \Delta e^\pi &= -|\nabla^\pi(\partial^\pi w)|^2 - 2\delta \langle *d^{\nabla^\pi} \partial^\pi w, * \partial^\pi w \rangle + 2|\delta^{\nabla^\pi} \partial^\pi w|^2 \\ &\quad - \langle K^\pi(dw, dw) \partial^\pi w, \partial^\pi w \rangle - R |\partial^\pi w|^2 \end{aligned}$$

where R is the Gaussian curvature of the given Kähler metric h on (Σ, j) and K^π is the curvature tensor of the contact Hermitian connection ∇^π .

It turns out that to establish the geometric analysis necessary for the study of associated moduli space, one needs to augment the equation $\bar{\partial}^\pi w = 0$ by

$$d(w^* \lambda \circ j) = 0. \tag{1.2}$$

Definition 1.5 (Contact instanton). Let Σ be as above. We call a pair (j, w) of j a complex structure on Σ and a map $w : \dot{\Sigma} \rightarrow Q$ a *contact instanton* if they satisfy

$$\bar{\partial}^\pi w = 0, \quad d(w^* \lambda \circ j) = 0. \tag{1.3}$$

We would like to point out that the system (1.3) (for a fixed j) forms an elliptic system, which is a natural elliptic twisting of the Cauchy-Riemann equation $\bar{\partial}^\pi w = 0$ then. (We refer to [Oh2] for the elaboration of this point of view.)

In hindsight, the more common twisting of $\bar{\partial}^\pi w = 0$ initiated by Hofer [H1] has been the twisting to the pseudoholomorphic curve system (1.1) of the pair (a, w) through the symplectization, *when $w^* \lambda \circ j$ is assumed to be exact*, where $a : \Sigma \rightarrow \mathbb{R}$ is an auxiliary potential function satisfying $w^* \lambda \circ j = da$. In this regard, the twisting (1.3) may be more natural in that it does not introduce additional auxiliary variable a , which is demonstrated by the a priori elliptic estimates and the exponential convergence results near the punctures established in this paper and sequels [OW2], [Oh2].

Another point which may be worthwhile to point out is that while the first part of the equation involves the first derivatives, the second part of the equation involves the second derivatives. Therefore it is not enough to have $W^{2,2}$ a priori estimate to get a classical solution out of a weak solution, and hence establishing at least $W^{3,2}$ coercive estimate is crucial to start the bootstrapping arguments.

Then we first establish the following a priori $W^{2,2}$ -estimates for such a map.

Theorem 1.6. *Let (Σ, j) be a closed Riemann surface. Suppose w satisfies (1.3) on Σ and $|dw| \in L^2 \cap L^4$ on Σ . Then*

$$\begin{aligned} \int_{\Sigma} |\nabla dw|^2 &\leq \frac{1}{2}(C_1^2 + 1) \|\partial^\pi w\|_{L^4}^4 - 2 \min R \|\partial^\pi w\|_{L^2; \supp R}^2 + \frac{3}{2} C_1^2 \|w^* \lambda\|_{L^4}^2 \|\partial^\pi w\|_{L^4}^2 \\ &\quad + 2 \max |K^\pi| (\|w^* \lambda\|_{L^4}^2 \|\partial^\pi w\|_{L^4}^2 + \|\partial^\pi w\|_{L^4}^4). \end{aligned}$$

Here K^π is the curvature of ∇^π , R the Gaussian curvature of the metric h on Σ and

$$C_1 = \|\mathcal{L}_{X_\lambda} J\|_{C^0}.$$

An immediate corollary of this theorem and the standard interpolation inequality of the L^p -norms is the following $W^{2,2}$ -coercive estimates.

Corollary 1.7. *Let Σ and w be as above. Suppose w satisfies (1.3) on Σ and $|dw| \in L^2 \cap L^4$ on Σ . Then there exist uniform constants C_3, C_4 depending only on $\|K^\pi\|_{C^0}$, $\|R\|_{C^0}$ and $\|\mathcal{L}_{X_\lambda} J\|_{C^0}$ but independent of w such that*

$$\|dw\|_{W^{1,2}}^2 \leq C_3 \|dw\|_{L^4}^4 + C_4 \|dw\|_{L^2}^2.$$

We also need to have the following local version of the main estimates which is an important ingredient for the local regularity and the bubbling-off analysis.

Theorem 1.8. *Let $D = D^2(1)$ be the unit disc and let $D' \subset \overline{D}' \subset D$ be another smaller disc. Then there exists $\epsilon > 0$ such that if $w : D \rightarrow Q$ is a smooth map satisfying $\bar{\partial}^\pi w = 0$ and $d(w^* \lambda \circ j) = 0$, and $E_{(\lambda, J)}(w) < \epsilon$,*

$$\|dw\|_{1,2;D'}^2 \leq C_5 \|dw\|_{4;D}^4 + C_6 \|dw\|_{2;D}^2$$

for some constants C_5, C_6 which depend only on $\|K^\pi\|_{C^0}$, $\|R\|_{C^0}$ and $\|\mathcal{L}_{X_\lambda} J\|_{C^0}$.

For the punctured Riemann surface $\dot{\Sigma}$, one need to put suitable asymptotic conditions in terms of the cylindrical metric and its associated isothermal coordinates denoted by (τ, t) . For this purpose, we need to impose asymptotic convergence behavior of the following L^2 -integral function

$$f(\tau) = \frac{1}{2} \int_{S^1} |d^\pi w|^2(\tau, t) dt \quad (1.4)$$

and the one-form $w^* \lambda = a_1 d\tau + a_2 dt$, besides requiring $|d^\pi w| \in L^2 \cap L^4$.

Theorem 1.9. *Let (Σ, j) be a closed Riemann surface with a finite number of marked points $\{r_1, \dots, r_k\}$. Denote by $\dot{\Sigma}$ the associated punctured Riemann surface with a Kähler metric h on (Σ, j) which is cylindrical near the puncture. Let f_ℓ be the function defined as above associated to the ℓ -th puncture r_ℓ . Suppose w satisfies (1.3) on $\dot{\Sigma}$ and $|d^\pi w| \in L^2 \cap L^4$ and $|w^* \lambda| \in C^0$ on $\dot{\Sigma}$*

$$\begin{aligned} \lim_{\tau \rightarrow \infty} a_1 &= a, & \lim_{\tau \rightarrow \infty} a_2 &= T \\ \lim_{\tau \rightarrow \infty} f_\ell(\tau) &= 0, & \lim_{\tau \rightarrow \infty} f'_\ell(\tau) &= 0 \end{aligned} \quad (1.5)$$

for all $\ell = 1, \dots, k$. Then

$$\begin{aligned} \int_{\dot{\Sigma}} |\nabla dw|^2 &\leq \frac{1}{2}(C_1^2 + 1) \|\partial^\pi w\|_{L^4}^4 - 2 \min R \|\partial^\pi w\|_{L^2; \supp R}^2 + \frac{3}{2} C_1^2 \|w^* \lambda\|_{C^0}^2 \|\partial^\pi w\|_{L^4}^2 \\ &\quad + 2 \max |K^\pi| (\|w^* \lambda\|_{C^0}^2 \|\partial^\pi w\|_{L^2}^2 + \|\partial^\pi w\|_{L^4}^4). \end{aligned}$$

Here K^π is the curvature of ∇^π , R the Gaussian curvature of the metric h on Σ and

$$C_1 = \|\mathcal{L}_{X_\lambda} J\|_{C^0}.$$

We would like to remark that the asymptotic boundary conditions imposed in this theorem will be also established in part 2 under the Hypothesis 6.2 together with nondegeneracy of the asymptotic Reeb orbits obtained from subsequence convergence theorem, Theorem 6.3.

Once this $W^{2,2}$ -estimate is proved, further differentiations of ∇dw and inductive alternating bootstrapping give rise to the all the higher regularity estimates too. We just state the more nontrivial punctured case here.

Theorem 1.10. *Let (Σ, j) and w be as above. Then if w satisfies (1.3) on $\dot{\Sigma}$ and $|d^\pi w| \in L^2 \cap L^4$ and $|w^* \lambda| \in C^0$ on $\dot{\Sigma}$, and (8.10), then*

$$\int_{\dot{\Sigma}} |(\nabla)^{k+1}(dw)|^2 \leq \int_{\dot{\Sigma}} J'_k(d^\pi w, w^* \lambda).$$

Here J'_{k+1} a polynomial function of the norms of the covariant derivatives of $d^\pi w$, $w^* \lambda$ up to $0, \dots, k$ with degree at most $2k + 4$ whose coefficients depend on

$$\|R\|_{C^k}, \|K^\pi\|_{C^k}, \|\mathcal{L}_{X_\lambda} J\|_{C^k}, \|w^* \lambda\|_{C^0}.$$

In particular,

$$\|dw\|_{W^{k+1,2}} \leq C_k(\|dw\|_{L^2}, \|dw\|_{L^4})$$

for a similar polynomial function $C_k = C_k(s, t)$.

The local version of the estimate also holds.

Theorem 1.11. *Let $D = D^2(1)$ be the unit disc. There exist $C_{5;k}, C_{6;k} > 0$ depending only on $D' \subset \overline{D}' \subset D$ and on $\|K^\pi\|_{C^k}, \|\mathcal{L}_{X_\lambda} J\|_{C^k}$ and $\|R\|_{C^k;D}$ but independent of w such that for any smooth map $w : D \rightarrow Q$ satisfying*

$$\overline{\partial}^\pi w = 0, \quad d(w^* \lambda \circ j) = 0,$$

$$\|dw\|_{k+1,2;D'} \leq C_{k;D,D'}(\|dw\|_{2;D}, \|dw\|_{4;D})$$

for a polynomial function $C_{k;D,D'}(s, t)$ of s, t up to $0, \dots, k$ of degree at most $2k + 4$ depending also on D', D . In particular, any weak solution of (1.3) in $W^{1,4}$ automatically becomes a classical solution of (1.3).

We refer to Theorem 8.4 and 5.8 and discussions around them for further expounding of these estimates.

Next, we examine the behavior of the contact instanton map near each puncture of a punctured Riemann surface. In this regard, it is crucial to prove some asymptotic convergence result to a closed Reeb orbit under a suitable finite energy hypothesis. For this purpose, we prove the following convergence result. We refer readers to Theorem 6.3 for more precise statements.

Theorem 1.12. *Let w be any contact instanton on $[0, \infty) \times S^1$ with finite π -harmonic energy*

$$E^\pi(w) := \frac{1}{2} \int_{[0, \infty) \times S^1} |d^\pi w|^2 < \infty, \quad \|dw\|_{C^0; [0, \infty) \times S^1} < \infty. \quad (1.6)$$

Then for any sequence $\tau_k \rightarrow \infty$, there exists a subsequence, still denoted by τ_k , and a massless instanton $w_\infty(\tau, t) = \gamma(a\tau + Tt)$ (i.e., $E^\pi(w_\infty) = 0$) on the cylinder $\mathbb{R} \times S^1$ along a closed Reeb orbit γ with period T such that

$$\lim_{k \rightarrow \infty} w(\tau_k + \tau, t) = w_\infty(\tau, t)$$

uniformly on $[-K, K] \times S^1$ for any given $K \geq 0$, where γ is a T -periodic orbit of X_λ . Here T and a are determined by

$$T = \int_{[0, \infty) \times S^1} |d^\pi w|^2 + \int_{S^1} w(0, \cdot)^* \lambda \quad (1.7)$$

$$a := - \int_{S^1} w(0, \cdot)^* \lambda \circ j = \int_{S^1} \lambda \left(\frac{\partial w}{\partial \tau}(0, t) \right) dt. \quad (1.8)$$

In particular, when $a = 0$, the limiting instanton w_∞ is translation invariant and so $w_\infty(\tau, t) \equiv \gamma(Tt)$, and the convergence is uniform and exponentially fast.

In the context of symplectization, which roughly corresponds to the exact case of $a = 0$, this subsequence convergence theorem can be derived from Hofer's subsequence convergence result proved in [H1].

When (γ, T) is a nondegenerate Reeb orbit, the limit z does not depend on the choice of subsequence and the convergence is exponentially fast, we also establish exponential decay estimates for the nondegenerate limiting Reeb orbit (and its Morse-Bott analog in a sequel [OW2]), whose proof is essentially different from that of [HWZ1, HWZ2], even for the exact context. Our proof follows canonical pedestrian tensorial calculations associated to the pair of connections, although it requires some ingenuity of combinatorics of tensor calculations. Our tensor calculations also clarify the geometry behind Hofer-Wysocki-Zehnder's coordinate calculations involved in their study of exponential decay estimates even for the pseudoholomorphic curves on the symplectization of contact manifolds (or on the symplectic manifolds with cylindrical ends). We would like to recall that this on-shell exponential decay estimate is one of the crucial analytical ingredients in setting up the off-shell functional analytic framework for the study of moduli space of contact instantons on the punctured Riemann surfaces residing in the contact manifold Q .

As for the methodology of the proof of exponential convergence, one could say that Hofer-Wysocki-Zehnder's derivation of exponential decay estimates is the *second-order method* while ours is the *third-order one* in that the former relies on the study of the second derivative of the integral

$$\int_{S^1} |u(\tau, \cdot)|^2 dt$$

through coordinate calculation using *special coordinates* for $w = (\theta, u) \in S^1 \times \mathbb{R}^{2n}$, while ours relies on the study of the second derivative of the canonical integral

$$\int_{S^1} |d^\pi w|^2 dt$$

through the invariant tensorial calculation using the *special connections*. We are very keen to make our C^k -exponential estimates (for $k \geq 1$) use only C^{k-1} -exponential estimates and the standard bootstrap argument. For this purpose, we need to derive a precise geometric formula of the Laplacian of the energy density function and the second covariant differential of w . We also perform several times of integration by parts which though require some ingenuity in combinatorics

of tensor calculations, will remove all the second order derivative terms of the map w after integrating over $t \in S^1$.

Further systematic study of the moduli space of contact instantons in the point of view of gauged sigma model is given in [Oh2].

Some historical recount on the equation (1.3) should be in order. After the original version of the present paper was posted in the arXiv e-print, we were informed that the equation (1.3) was first mentioned by Hofer in p.698 of [H2]. Hofer also described its possible usage in the following form

$$\begin{cases} \bar{\partial}^\pi w = 0 \\ w^* \lambda \circ j = da + \gamma \end{cases}$$

for a given *smooth* harmonic one-form γ defined on the *closed* Riemann surface Σ smoothly extending across the punctures. (See also [ACH]). In the language of the current paper, the second part of the equation forces *any w arising from a finite π -energy solution thereof will have vanishing ‘charge’ at the puncture where the charge is given by*

$$\int_{S^1} w(\tau, \cdot)^* \lambda \circ j$$

in the notation of current paper: any solution to their equation hence does not have spiral along the limiting Reeb orbit. The same form of the equation was also used by Abbas [Ab] to prove some existence result in relation to the open book decomposition in 3 dimension and seems to be a useful restriction for the purpose of exponential convergence result near the punctures. (See section 6 and 11 for relevant discussion, especially Remark 6.5.)

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2. CONTACT TRIAD CONNECTION AND CONTACT HERMITIAN CONNECTION

In this section, we recall the basic properties of the contact triad connection and the definition of the associated contact Hermitian connection of the Hermitian vector bundle $(\xi, d\lambda|_\xi, J)$.

Theorem 2.1 (Contact triad connection [OW1]). *Let (Q, λ, J) be any contact triad of contact manifold (Q, ξ) . Denote by*

$$g_\xi + \lambda \otimes \lambda =: g$$

the natural Riemannian metric on Q induced by (λ, J) , which we call a contact triad metric. Then there exists a unique affine connection ∇ that has the following properties:

- (1) ∇ is a Riemannian connection of the triad metric.
- (2) The torsion tensor of ∇ satisfies $T(X_\lambda, Y) = 0$ for all $Y \in TQ$.
- (3) $\nabla_{X_\lambda} X_\lambda = 0$ and $\nabla_Y X_\lambda \in \xi$, for $Y \in \xi$.

- (4) $\nabla^\pi := \pi\nabla|_\xi$ defines a Hermitian connection of the vector bundle $\xi \rightarrow Q$ with Hermitian structure $(d\lambda|_\xi, J)$.
 (5) The ξ projection of the torsion T , denoted by $T^\pi := \pi T$ satisfies the following properties:

$$T^\pi(JY, Y) = 0 \quad (2.1)$$

for all Y tangent to ξ .

- (6) For $Y \in \xi$,

$$\partial_Y^\nabla X_\lambda = \frac{1}{2}(\nabla_Y X_\lambda - J\nabla_{JY} X_\lambda) = 0.$$

We call ∇ the contact triad connection.

We would like to note that Axioms (4) and (5) are nothing but the properties of canonical connection on the tangent bundle of the (non-Hausdorff) almost Kähler manifold $(\widehat{Q}, \widehat{d\lambda}, \widehat{J}_\xi)$ of the leaf space of Reeb foliation of the contact form λ lifted to ξ . (As in the almost Kähler case [Ko], vanishing of $(1, 1)$ -component automatically gives rise to the vanishing of $(2, 0)$ -component of T . So in the presence of other axioms, Axiom (5) is equivalent to saying that T is of $(0, 2)$ -type. We refer to [OW1] for the details.)

On the other hand, Axioms (1), (2), (3) indicate this connection behaves like the Levi-Civita connection when the Reeb direction X_λ get involved. Axiom (6) is an extra condition to connect the information in ξ part and X_λ part, which is used to dramatically simplify our tensor calculations in the present paper (see Proposition 2.2).

The contact triad connection ∇ canonically induces a Hermitian connection for the Hermitian vector bundle (ξ, J, g_ξ) with $g_\xi = d\lambda(\cdot, J\cdot)|_\xi$. We denote this vector bundle connection by ∇^π and call it the *contact Hermitian connection*.

The following fundamental property of contact triad connection was proved in [OW1], which is one property, together with J -linearity of connection ∇ on ξ and its torsion property of being $(0, 2)$ -type, that greatly simplifies our tensorial calculations performed in the present paper.

Proposition 2.2 (Proposition 6.11 [OW1]). *Let ∇ be the contact triad connection. Then*

$$\nabla_Y X_\lambda = \frac{1}{2}(\mathcal{L}_{X_\lambda} J)JY$$

for any vector field Y on Q .

Part 1. A priori estimates and asymptotic convergence

In this part, we illustrate the effectiveness of the contact triad connection and its associated contact Hermitian connection in the tensorial calculations of the Laplacian of the ξ -component of the harmonic energy density functions. This computation is one of the essential steps of obtaining various a priori estimates needed in the study of regularity and compactness properties of solutions of geometric partial differential equations. We also derive an optimal form of the second covariant differentials of contact Cauchy-Riemann maps which we call the *fundamental equation*, and establish the basic asymptotic subsequence convergence to closed Reeb orbits at each puncture of punctured Riemann surface $\dot{\Sigma}$.

Using this optimal form of the second derivative and the study of perturbation results of eigenvalues, we carry out the inductive bootstrapping argument of

deriving the C^∞ -exponential estimates starting from the C^1 bound of w and the subsequence asymptotic convergence to a Reeb orbit in Part 2.

3. LINEARIZATION OPERATOR OF REEB ORBITS

Now we would like to study the linearization of the equation $\dot{x} = X_\lambda(x)$ along a closed Reeb orbit. The materials in this section are mostly standard and well-known results in contact geometry. See Appendix [ABW] for the exposition that is the closest to the one given in this section. Since our systematic usage of contact triad connection in the exposition gives rise to some explicit useful formulae occurring in this study which will be also important in our proof of exponential convergence, we provide precise statements and details of proofs of the results that will be relevant to the study of later sections.

Let γ be a closed Reeb orbit of period $T > 0$. In other words, $\gamma : \mathbb{R} \rightarrow Q$ is a periodic solution of $\dot{\gamma} = X_\lambda(\gamma)$ with period T , thus satisfying $\gamma(T) = \gamma(0)$.

Denote the Reeb flow $\phi^t := \phi_{X_\lambda}^t$ of the Reeb vector field X_λ , we can write $\gamma(t) = \phi^t(\gamma(0))$. In particular $p := \gamma(0)$ is a fixed point of the diffeomorphism ϕ^T when γ is a closed Reeb orbit of period T . We will call the pair (T, z) a Reeb orbit of period T instead for a such closed orbit γ of period T by writing $z(t) = \gamma(Tt)$ for a loop parameterized over the unit interval $S^1 = [0, 1]/\sim$. Since $\mathcal{L}_{X_\lambda}\lambda = 0$, the contact diffeomorphism ϕ^T canonically induces the isomorphism

$$\Psi_z := d\phi^T(p)|_{\xi_p} : \xi_p \rightarrow \xi_p$$

which is the linearized Poincaré return map ϕ^T restricted to ξ_p via the splitting $T_p Q = \xi_p \oplus \mathbb{R} \cdot \{X_\lambda(p)\}$.

Definition 3.1. We say a Reeb orbit with period T , (T, z) , is *nondegenerate* if the linearized return map $\Psi_z : \xi_p \rightarrow \xi_p$ with $p = z(0)$ has no eigenvalue 1.

Denote $\text{Cont}(Q, \xi)$ the set of contact one-forms with respect to the contact structure ξ and $\mathcal{L}(Q) = C^\infty(S^1, Q)$ the space of loops $z : S^1 = \mathbb{R}/\mathbb{Z} \rightarrow Q$.

Let $\mathcal{L}^{1,2}(Q)$ be the $W^{1,2}$ -completion of $\mathcal{L}(Q)$. We would like to consider some Banach vector bundle \mathcal{L} over the Banach manifold $(0, \infty) \times \mathcal{L}^{1,2}(Q) \times \text{Cont}(Q, \xi)$ whose fiber at (T, z, λ) is given by $L^2(z^*TQ)$. We consider the assignment

$$\Upsilon : (T, z, \lambda) \mapsto \dot{z} - T X_\lambda(z)$$

which is a section of \mathcal{L} . Then $(T, z, \lambda) \in \Upsilon^{-1}(0)$ if and only if there exists some Reeb orbit $\gamma : \mathbb{R} \rightarrow Q$ with period T , such that $z(\cdot) = \gamma(T\cdot)$.

We also denote $DX_\lambda : \Omega^0(\xi) \rightarrow \Omega^0(\xi)$ the covariant derivative of X_λ induced from the contact triad connection ∇ to highlight its aspect as a linear operator, whenever we feel convenient. The following derivation of the linearization of Υ is a routine exercise. (See Appendix [ABW] for a formula that is close to the current form.) Since it is not essential to our purpose in this paper, we omit its derivation only by stating the final result.

Lemma 3.2. *For any torsion free connection,*

$$d(T, z, \lambda)\Upsilon(a, Y, B) = \frac{DY}{dt} - TDX_\lambda(z)(Y) - aX_\lambda - T\delta_\lambda X_\lambda(B),$$

where $a \in \mathbb{R}$, $Y \in T_z \mathcal{L}^{1,2}(Q) = W^{1,2}(z^*TQ)$, $B \in T_\lambda \text{Cont}(Q, \xi)$ and the last term $\delta_\lambda X_\lambda$ is some linear operator.

We remark that the contact triad connection we use this paper is *not* torsion-free. However, when $(T, z, \lambda) \in \Upsilon^{-1}(0)$, i.e., $z(\cdot) = \gamma(T\cdot)$ for some γ which is a Reeb orbit with period T with respect to contact one-form λ , the torsion Axiom (2) in Definition 2.1 is already enough to derive Lemma 3.2. From now on, we use the contact triad connection through out this section. We recall the readers that the linearization at $(T, z, \lambda) \in \Upsilon^{-1}(0)$ actually doesn't depend of the choice of connections.

In this paper we only need to look at the linearization restricted to subspace $W^{1,2}(z^*\xi)$ for fixed (T, λ) . Denote the corresponding operator by

$$\Upsilon_{T,\lambda} = \Upsilon(T, \cdot, \lambda).$$

We have the following characterization of the nondegeneracy condition.

Proposition 3.3. *A closed Reeb orbit γ with period T is nondegenerate if and only if the ξ projection of the linearization restricted to $W^{1,2}(z^*\xi)$, i.e.,*

$$d_z^\pi \Upsilon_{T,\lambda}|_{W^{1,2}(z^*\xi)} := \pi d_z \Upsilon_{T,\lambda}|_{W^{1,2}(z^*\xi)} : W^{1,2}(z^*\xi) \rightarrow L^2(z^*\xi)$$

is surjective, where $z(\cdot) := \gamma(T\cdot) : S^1 \rightarrow Q$.

The rest of the section will be occupied by the proof of this proposition.

From Lemma 3.2 and Proposition 2.2, we compute

$$\begin{aligned} d_z^\pi \Upsilon_{T,\lambda}|_{W^{1,2}(z^*\xi)}(\zeta) &= \pi \frac{D\zeta}{dt} - T \cdot \pi DX_\lambda(z)(\zeta) \\ &= \frac{D^\pi \zeta}{dt} - \frac{T}{2}(\mathcal{L}_{X_\lambda} J)J\zeta. \end{aligned}$$

Since from Axiom (3) of Definition 2.1, the image of $d_z \Upsilon_{T,\lambda}|_{W^{1,2}(z^*\xi)}$ automatically in ξ .

Lemma 3.4. *Let $DX_\lambda(z) = \nabla_{(\cdot)} X_\lambda : z^*TQ \rightarrow z^*TQ$ be the covariant derivative of X_λ with respect to the pull-back connection $z^*\nabla$ of the contact triad connection. Consider a Reeb orbit (T, z) i.e., a map $z : S^1 \rightarrow Q$ satisfying $\dot{z} = TX_\lambda(z)$ with $z(1) = z(0)$. Then*

$$DX_\lambda(z)(Y) = \frac{1}{2}(\mathcal{L}_{X_\lambda} J(z))J(z)Y$$

*for any section $Y \in \Omega^0(z^*TQ)$.*

Proof. By definition, we have

$$DX_\lambda(z)(Y) = \nabla_Y X_\lambda$$

and then apply Proposition 2.2, which proves the equality. \square

We recall the following symmetry property from [OW1].

Lemma 3.5 (Lemma 6.2 [Bl], Lemma 5.2 [OW1]). *Both $(\mathcal{L}_{X_\lambda} J)J$ and $\mathcal{L}_{X_\lambda} J$ are pointwise symmetric with respect to the triad metric of (Q, λ, J) .*

Combining the above discussion, we have derived

Proposition 3.6. *The linear operator*

$$Jd_z^\pi \Upsilon_{T,\lambda} = J\pi \left(\frac{D}{dt} - DX_\lambda(z) \right) = J \frac{D^\pi}{dt} - \frac{T}{2}(\mathcal{L}_{X_\lambda} J) : L^2(z^*\xi) \rightarrow L^2(z^*\xi)$$

is a self-adjoint operator. In particular, we obtain

$$\text{Index } d_z^\pi \Upsilon_{T,\lambda} = \text{Index } Jd_z^\pi \Upsilon_{T,\lambda} = 0. \quad (3.1)$$

Finally we are ready to prove the above analytic characterization of the non-degeneracy. By Proposition 3.6, the surjectivity of $d_z^\pi \Upsilon_{T,\lambda}$ is equivalent to the injectivity of the operator. In fact, we prove the following characterization of kernel elements of the linearization map $d_z^\pi \Upsilon_{T,\lambda}$ in terms of the eigenvectors of the linear map $\Psi_p : \xi_p \rightarrow \xi_p$ where $\Psi_p = d\phi^T(p)|_{\xi_p}$.

Proposition 3.7. *Let $p = z(0)$ be a fixed point of $\phi^T : Q \rightarrow Q$ lying in the given Reeb orbit (T, z) . Then there exists a one-one correspondence*

$$v \in \xi_p \mapsto \eta; \quad \eta(t) := d\phi^{tT}(v), \quad t \in [0, 1]$$

between the set of eigenvectors v of $\Psi_p = d\phi^T|_{\xi_p} : \xi_p \rightarrow \xi_p$ with eigenvalue 1 and the set of solutions η to $\frac{D\eta}{dt} - \frac{T}{2}(\mathcal{L}_{X_\lambda} J)\eta = 0$.

Proof. Recall that any closed Reeb orbit of period T has the form $z(t) = \phi^{tT}(p)$ for a fixed point p of ϕ^{tT} .

Suppose η is a solution to $0 = d_z^\pi \Upsilon_{T,\lambda}(\eta) = \frac{D\eta}{dt} - \frac{T}{2}(\mathcal{L}_{X_\lambda} J)J\eta$. We consider the one-parameter family

$$v(t) = (d\phi^{tT})^{-1}(\eta(t))$$

of tangent vectors at $p \in Q$, and so $\eta(t) = d\phi^{tT}(v(t))$. We compute $\nabla_t \eta(t)$ by considering the map $\Gamma(s, t) = \phi^{tT}(\alpha(s, t))$ such that $\alpha(0, t) \equiv p$ and $\frac{\partial}{\partial s}\big|_{s=0} \alpha(s, t) = v(t)$. Then we compute

$$\frac{\partial \Gamma}{\partial s} = d\phi^{tT} \left(\frac{\partial \alpha}{\partial s} \right), \quad \frac{\partial \Gamma}{\partial t}(s, t) = TX_\lambda(\Gamma(s, t)) + d\phi^{tT} \left(\frac{\partial \alpha}{\partial t}(s, t) \right)$$

and so

$$\begin{aligned} \nabla_t \eta &= \frac{D}{dt} \frac{\partial \Gamma}{\partial s} \bigg|_{s=0} = \frac{D}{ds} \frac{\partial \Gamma}{\partial t} \bigg|_{s=0} \\ &= T \frac{D}{ds} (X_\lambda(\Gamma(s, t))) \bigg|_{s=0} + d\phi^{tT} \left(\frac{D}{ds} \bigg|_{s=0} \frac{\partial \alpha}{\partial t}(s, t) \right) \\ &= T \frac{D}{ds} (X_\lambda(\Gamma(s, t))) \bigg|_{s=0} + d\phi^{tT} \left(\frac{D}{\partial t} \frac{\partial \alpha}{\partial s} \bigg|_{s=0}(s, t) \right) \\ &= T \frac{D}{ds} (X_\lambda(\Gamma(s, t))) \bigg|_{s=0} + d\phi^{tT} (v'(t)). \end{aligned}$$

Here the second and the fourth equalities follow from the torsion property of the triad connection

$$T \left(\frac{\partial \Gamma}{\partial t} \bigg|_{s=0}, \frac{\partial \Gamma}{\partial s} \bigg|_{s=0} \right) = T(d\phi^{tT} v(t), X_\lambda(\phi^{tT}(p))) = 0.$$

The first term of the farthest right becomes

$$T \frac{D}{ds} (X_\lambda(\Gamma(s, t))) \bigg|_{s=0} = \frac{T}{2}(\mathcal{L}_{X_\lambda} J)J\eta.$$

Therefore we have derived

$$v'(t) = (d\phi^{tT})^{-1} \left(\nabla_t \eta - \frac{T}{2}(\mathcal{L}_{X_\lambda} J)J\eta \right) = 0.$$

by the hypothesis that η satisfies the equation $\frac{D\eta}{dt} - \frac{T}{2}(\mathcal{L}_{X_\lambda} J)J\eta = 0$. Therefore we have

$$v(1) = v(0), \text{ i.e., } (d\phi^T)^{-1}(\eta(1)) = \eta(0).$$

Since $\eta(0) = \eta(1)$, it implies that $J\eta(0)$ is an eigenvector of eigenvalue 1 if $\eta(0) \neq 0$.

Conversely suppose that v is an eigenvector of $\phi^{tT} : \xi_p \rightarrow \xi_p$. Then the above computation of v' applied to constant function $v(t) \equiv v$ proves that the vector field $t \mapsto d\phi^{tT}(v)$ satisfies $\nabla_t^\pi \eta - \frac{T}{2}(\mathcal{L}_{X_\lambda} J)J\eta = 0$. This finishes the proof. \square

This proposition in particular finishes the proof of the statement that $d_z^\pi \Upsilon_{T,\lambda}|_{W^{1,2}(z^*\xi)}$ is surjective if and only if $\Psi_\gamma = d\phi^T|_{\xi_p}$ has an eigenvalue 1 and so finish the proof of Proposition 3.3.

4. TENSORIAL CALCULATIONS FOR GEOMETRIC ENERGY DENSITY FUNCTION

From this section, we will use the contact Hermitian connection ∇^π for the hermitian bundle ξ over Q and the triad connection ∇ on Q to do the calculation. First, we combine the pull-back connection on $w^*\xi$, again denoted by ∇^π , and the Hermitian connection of the Riemann surface (Σ, j, h) . We get a connection on $T^*\Sigma \otimes w^*\xi$ which we still denoted by ∇^π .

Fix a Kähler metric h on (Σ, j) . The norm $|dw|$ of the map

$$dw : (T\Sigma, h) \rightarrow (TQ, g)$$

with respect to the metric g is defined by

$$|dw|_g^2 := \sum_{i=1}^2 |dw(e_i)|_g^2,$$

where $\{e_1, e_2\}$ is an orthonormal frame of $T\Sigma$ with respect to h .

The following off-shell formulae are immediate consequences of the compatibility of J to $d\lambda$ on ξ .

Proposition 4.1. *Denote $g_J = d\lambda(\cdot, J\cdot)|_\xi$ and the associated norm by $|\cdot| = |\cdot|_J$. Fix a Hermitian metric h of (Σ, j) , and consider a smooth map $u : \Sigma \rightarrow M$. Then we have*

- (1) $|d^\pi w|^2 = |\partial^\pi w|^2 + |\bar{\partial}^\pi w|^2$,
- (2) $2w^*d\lambda = (-|\bar{\partial}^\pi w|^2 + |\partial^\pi w|^2)dA$ where dA is the area form of the metric h on Σ .
- (3) $w^*\lambda \wedge w^*\lambda \circ j = -|w^*\lambda|^2 dA$
- (4) $|\nabla w^*\lambda|^2 = |dw^*\lambda|^2 + |\delta w^*\lambda|^2$.

In particular, if $\bar{\partial}^\pi w = 0$, then

$$|d^\pi w|^2 = |\partial^\pi w|^2, \quad w^*d\lambda = \frac{1}{2}|d^\pi w|^2 dA \quad (4.1)$$

Proof. The proofs of (1), (2) are exactly the same as the case of pseudoholomorphic maps in symplectic manifolds with replacement of dw by $d^\pi w$ and the symplectic form by $d\lambda$ and so omitted. (See e.g., Proposition 7.19 [Oh1] for the statements and their proofs in the symplectic case corresponding the statements (1), (2) here.) Statement (3) follows from the identity $w^*\lambda \circ j = - * w^*\lambda$ and by definition of the Hodge star operator, and then (4) is nothing but the Gårding's equality. \square

We denote by d^{∇^π} the skew-symmetrization of covariant derivative ∇^π given by

$$d^{\nabla^\pi}(\alpha)(\xi_1, \xi_2) = (\nabla_{\xi_1}^\pi \alpha)(\xi_2) - (\nabla_{\xi_2}^\pi \alpha)(\xi_1)$$

where $\xi_1, \xi_2 \in T\Sigma$.

It defines an operator

$$d^{\nabla^\pi} : \Omega^1(w^*\xi) \rightarrow \Omega^2(w^*\xi).$$

We denote by

$$\Delta^\pi = \delta^{\nabla^\pi} d^{\nabla^\pi} + d^{\nabla^\pi} \delta^{\nabla^\pi}$$

the Hodge Laplacian acting on one-forms, where δ^{∇^π} is the formal adjoint of d^{∇^π} . Then we have the following Weitzenböck formula

$$\Delta^\pi \beta = (\nabla^\pi)^* \nabla^\pi \beta + R dw + K^\pi(dw, dw) \beta \quad (4.2)$$

in general where R is the Gaussian curvature of the surface (Σ, j) and K^π is the curvature of the connection ∇^π on the vector bundle $\xi \rightarrow Q$.

On the flat cylinder, we have $R \equiv 0$, and so the equation further simplifies to

$$\Delta^\pi \beta = (\nabla^\pi)^* \nabla^\pi \beta + K^\pi(dw, dw) \beta.$$

In the remaining section, considering $\beta = \partial^\pi w := \pi \partial w$ as a one-form in $\Omega^1(w^* TQ)$, we will compute $\Delta^\pi(\partial^\pi w)$ for a contact Cauchy-Riemann map, i.e., a map satisfying $\bar{\partial}^\pi w = 0$.

We start with the following lemma. Recall we denote by $\Pi : TQ \rightarrow TQ$ the idempotent associated to the projection $\pi : TQ \rightarrow \xi$, i.e., the endomorphism satisfying

$$\Pi^2 = \Pi, \quad \text{Im } \Pi = \xi, \quad \ker \Pi = \mathbb{R}\{X_\lambda\}.$$

Lemma 4.2. *Let $w : \Sigma \rightarrow Q$ be any smooth map. Denote $d^\pi w = \pi dw \in \Omega^1(w^* \xi)$. As a two-form with value in $w^* \xi$, $d^{\nabla^\pi}(d^\pi w)$ has the expression*

$$d^{\nabla^\pi}(d^\pi w) = T^\pi(\Pi dw, \Pi dw) + w^* \lambda \wedge \left(\frac{1}{2} (\mathcal{L}_{X_\lambda} J) J d^\pi w \right) \quad (4.3)$$

where T is the torsion tensor of ∇ .

Proof. For given $\xi_1, \xi_2 \in \Gamma(T\Sigma)$, we evaluate

$$\begin{aligned} d^{\nabla^\pi}(d^\pi w)(\xi_1, \xi_2) &= d^{\nabla^\pi}(\pi dw)(\xi_1, \xi_2) \\ &= (\nabla_{\xi_1}^\pi(\pi dw))(\xi_2) - (\nabla_{\xi_2}^\pi(\pi dw))(\xi_1) \\ &= (\nabla_{\xi_1}^\pi(\pi dw(\xi_2)) - \pi dw(\nabla_{\xi_1} \xi_2)) - (\nabla_{\xi_2}^\pi(\pi dw(\xi_1)) - \pi dw(\nabla_{\xi_2} \xi_1)) \\ &= \pi \left((\nabla_{\xi_1}(dw(\xi_2)) - \nabla_{\xi_1}(\lambda(dw(\xi_2))X_\lambda)) - (\nabla_{\xi_2}(dw(\xi_1)) - \nabla_{\xi_2}(\lambda(dw(\xi_1))X_\lambda)) \right. \\ &\quad \left. - dw(\nabla_{\xi_1} \xi_2 - \nabla_{\xi_2} \xi_1) \right) \\ &= \pi \left(\nabla_{\xi_1}(dw(\xi_2)) - \nabla_{\xi_2}(dw(\xi_1)) - [dw(\xi_1), dw(\xi_2)] \right. \\ &\quad \left. - \nabla_{\xi_1}(\lambda(dw(\xi_2))X_\lambda) + \nabla_{\xi_2}(\lambda(dw(\xi_1))X_\lambda) \right) \\ &= \pi \left(T(dw(\xi_1), dw(\xi_2)) - \lambda(dw(\xi_2))\nabla_{\xi_1} X_\lambda - \xi_1[\lambda(dw(\xi_2))]X_\lambda \right. \\ &\quad \left. + \lambda(dw(\xi_1))\nabla_{\xi_2} X_\lambda + \xi_2[\lambda(dw(\xi_1))]X_\lambda \right) \\ &= \pi \left(T(dw(\xi_1), dw(\xi_2)) - \lambda(dw(\xi_2))\nabla_{\xi_1} X_\lambda + \lambda(dw(\xi_1))\nabla_{\xi_2} X_\lambda \right) \\ &= T^\pi(\Pi dw(\xi_1), \Pi dw(\xi_2)) \\ &\quad + \frac{1}{2} \lambda(dw(\xi_2))J(\mathcal{L}_{X_\lambda} J)\pi dw(\xi_1) - \frac{1}{2} \lambda(dw(\xi_1))J(\mathcal{L}_{X_\lambda} J)\pi dw(\xi_2) \\ &= T^\pi(\Pi dw(\xi_1), \Pi dw(\xi_2)) \\ &\quad - \frac{1}{2} \lambda(dw(\xi_2))(\mathcal{L}_{X_\lambda} J)J\pi dw(\xi_1) + \frac{1}{2} \lambda(dw(\xi_1))(\mathcal{L}_{X_\lambda} J)J\pi dw(\xi_2) \end{aligned}$$

Here for the last second equality, we use Proposition 2.2 and Axiom (3) for this connection. We rewrite the above result as

$$d^{\nabla^\pi}(d^\pi w) = T^\pi(\Pi dw, \Pi dw) + w^* \lambda \wedge \left(\frac{1}{2}(\mathcal{L}_{X_\lambda} J) J d^\pi w \right)$$

for any w . We have finished the proof. \square

Now let w be a solution to

$$\bar{\partial}^\pi w = 0. \quad (4.4)$$

Then we have

$$\begin{aligned} d^\pi w &= \partial^\pi w \\ J \partial^\pi w &= \partial^\pi w \circ j. \end{aligned} \quad (4.5)$$

As an immediate corollary of the previous lemma applied to the solution w , we derive the following theorem of the fundamental equation. This is the contact analog of [Oh1, Proposition 7.27].

Theorem 4.3 (Fundamental equation). *Let w be a contact Cauchy-Riemann map, i.e., a solution of (4.4). Then*

$$\begin{aligned} d^{\nabla^\pi}(d^\pi w) &= d^{\nabla^\pi}(\partial^\pi w) \\ &= -w^* \lambda \circ j \wedge \left(\frac{1}{2}(\mathcal{L}_{X_\lambda} J) \partial^\pi w \right). \end{aligned} \quad (4.6)$$

Proof. The first equality follows since $d^\pi w = \partial^\pi w$ for the solution w . Also, it follows

$$T^\pi(\Pi dw, \Pi dw) = T^\pi(\partial^\pi w, \partial^\pi w) = 0$$

since the torsion $T^\pi|_\xi$ is of $(0, 2)$ -type, in particular, has vanishing $(1, 1)$ -component. Further we write (4.3) as

$$\begin{aligned} d^{\nabla^\pi}(d^\pi w) &= w^* \lambda \wedge \left(\frac{1}{2}(\mathcal{L}_{X_\lambda} J) J \partial^\pi w \right) \\ &= w^* \lambda \wedge \left(\frac{1}{2}(\mathcal{L}_{X_\lambda} J) \partial^\pi w \right) \circ j \\ &= -w^* \lambda \circ j \wedge \left(\frac{1}{2}(\mathcal{L}_{X_\lambda} J) \partial^\pi w \right), \end{aligned}$$

where we use the identity $J \partial^\pi w = \partial^\pi w \circ j$. This finishes the proof. \square

Now we compute

$$\Delta^\pi(\partial^\pi w) = \delta^{\nabla^\pi} d^{\nabla^\pi}(\partial^\pi w) + d^{\nabla^\pi} \delta^{\nabla^\pi}(\partial^\pi w).$$

Recall the Hodge dual formula for dimension 2 spaces

$$\delta^{\nabla^\pi} = - * d^{\nabla^\pi} *.$$

We have the following lemma which can greatly simplify our calculation of deriving the energy density formulae. This lemma is an interpretation of the metric property of the connection for forms. We postpone its proof till the appendix, Section 9.

Lemma 4.4. *Assume α is a zero-form in $\Omega^0(w^* \xi)$ and β is a one-form in $\Omega^1(w^* \xi)$. $\langle \cdot, \cdot \rangle$ is the inner production on $w^* \xi$ introduced from the metric of Q . Then we have*

$$\langle d^{\nabla^\pi} \alpha, \beta \rangle - \langle \alpha, \delta^{\nabla^\pi} \beta \rangle = -\delta \langle \alpha, \beta \rangle.$$

The following lemma is another useful formula whose proof is a straightforward calculation and will be given in the appendix.

Lemma 4.5. *For any connection ∇ and vector-valued one-form α ,*

$$|\nabla\alpha|^2 = |d^\nabla\alpha|^2 + |\delta^\nabla\alpha|^2 - *(\nabla\alpha \wedge \nabla_{j(\cdot)}\alpha).$$

In the cylindrical coordinates, it has the expression

$$|\nabla\alpha|^2 = |d^\nabla\alpha|^2 + |\delta^\nabla\alpha|^2 + 2\langle \nabla_\tau\eta, \nabla_t\zeta \rangle - 2\langle \nabla_t\eta, \nabla_\tau\zeta \rangle.$$

We would like to point out that the last term $*(\nabla\alpha \wedge \nabla_{j(\cdot)}\alpha)$ vanishes for real valued (or any line bundle valued) forms but does not for general higher rank vector bundle valued forms.

Before we use these formulae to compute $\langle \Delta^\pi \partial^\pi w, \partial^\pi w \rangle$, we first state the following lemma which is due to (4.5) and J -compatibility of the metric. We also remark that this lemma holds only for the connection ∇^π which is J -linear.

Lemma 4.6. *For any smooth map w , we have*

$$\langle d^{\nabla^\pi} \delta^{\nabla^\pi} \partial^\pi w, \partial^\pi w \rangle = \langle \delta^{\nabla^\pi} d^{\nabla^\pi} \partial^\pi w, \partial^\pi w \rangle.$$

As a consequence,

$$\langle \Delta^\pi \partial^\pi w, \partial^\pi w \rangle = 2\langle \delta^{\nabla^\pi} d^{\nabla^\pi} \partial^\pi w, \partial^\pi w \rangle. \quad (4.7)$$

Proof.

$$\begin{aligned} \langle \delta^{\nabla^\pi} d^{\nabla^\pi} \partial^\pi w, \partial^\pi w \rangle &= -\langle *d^{\nabla^\pi} *d^{\nabla^\pi} \partial^\pi w, \partial^\pi w \rangle \\ &= -\langle d^{\nabla^\pi} *d^{\nabla^\pi} \partial^\pi w, *\partial^\pi w \rangle \\ &= -\langle d^{\nabla^\pi} *d^{\nabla^\pi} \partial^\pi w, -\partial^\pi w \circ j \rangle \end{aligned} \quad (4.8)$$

$$\begin{aligned} &= \langle d^{\nabla^\pi} *d^{\nabla^\pi} \partial^\pi w, J\partial^\pi w \rangle \\ &= -\langle Jd^{\nabla^\pi} *d^{\nabla^\pi} \partial^\pi w, \partial^\pi w \rangle \\ &= -\langle d^{\nabla^\pi} *d^{\nabla^\pi} J\partial^\pi w, \partial^\pi w \rangle \end{aligned} \quad (4.9)$$

$$= -\langle d^{\nabla^\pi} *d^{\nabla^\pi} \partial^\pi w \circ j, \partial^\pi w \rangle \quad (4.10)$$

$$\begin{aligned} &= \langle d^{\nabla^\pi} *d^{\nabla^\pi} * \partial^\pi w, \partial^\pi w \rangle \\ &= \langle d^{\nabla^\pi} \delta^{\nabla^\pi} \partial^\pi w, \partial^\pi w \rangle. \end{aligned} \quad (4.11)$$

Here for (4.8) and (4.11), we use $*\alpha = -\alpha \circ j$ for any one-form α . For (4.9), we use the connection is J -linear. \square

Now we are ready to state the following lemma.

Lemma 4.7.

$$\langle \Delta^\pi \partial^\pi w, \partial^\pi w \rangle = -2\delta \langle *d^{\nabla^\pi} \partial^\pi w, *\partial^\pi w \rangle + 2|\delta^{\nabla^\pi} \partial^\pi w|^2.$$

Proof. Using (4.7) and Lemma 4.4 we compute

$$\begin{aligned}
\langle \Delta^\pi \partial^\pi w, \partial^\pi w \rangle &= 2\langle \delta^{\nabla^\pi} d^{\nabla^\pi} \partial^\pi w, \partial^\pi w \rangle \\
&= 2\langle - * d^{\nabla^\pi} * d^{\nabla^\pi} \partial^\pi w, \partial^\pi w \rangle \\
&= 2\langle *(- * d^{\nabla^\pi} * d^{\nabla^\pi} \partial^\pi w), * \partial^\pi w \rangle \\
&= 2\langle d^{\nabla^\pi} * d^{\nabla^\pi} \partial^\pi w, * \partial^\pi w \rangle \\
&= -2\delta\langle * d^{\nabla^\pi} \partial^\pi w, * \partial^\pi w \rangle + 2\langle * d^{\nabla^\pi} \partial^\pi w, \delta^{\nabla^\pi} * \partial^\pi w \rangle \\
&= -2\delta\langle * d^{\nabla^\pi} \partial^\pi w, * \partial^\pi w \rangle - 2\langle * d^{\nabla^\pi} \partial^\pi w, (* d^{\nabla^\pi} *) * \partial^\pi w \rangle \\
&= -2\delta\langle * d^{\nabla^\pi} \partial^\pi w, * \partial^\pi w \rangle + 2\langle * d^{\nabla^\pi} \partial^\pi w, * d^{\nabla^\pi} \partial^\pi w \rangle \\
&= -2\delta\langle * d^{\nabla^\pi} \partial^\pi w, * \partial^\pi w \rangle + 2|d^{\nabla^\pi} \partial^\pi w|^2 \\
&= -2\delta\langle * d^{\nabla^\pi} \partial^\pi w, * \partial^\pi w \rangle + 2|\delta^{\nabla^\pi} \partial^\pi w|^2.
\end{aligned}$$

For the last equality we use

$$|d^{\nabla^\pi} \partial^\pi w| = |\delta^{\nabla^\pi} \partial^\pi w| \quad (4.12)$$

which comes from the fact that

$$\begin{aligned}
\delta^{\nabla^\pi} \partial^\pi w &= - * d^{\nabla^\pi} * \partial^\pi w = * d^{\nabla^\pi} \partial^\pi w \circ j \\
&= * d^{\nabla^\pi} J \partial^\pi w = J * d^{\nabla^\pi} \partial^\pi w.
\end{aligned}$$

□

Now we consider the energy density function for a contact Cauchy-Riemann map w defined as

$$e^\pi := |d^\pi w|^2 = (|\pi dw(e_1)|^2 + |\pi dw(e_2)|^2)$$

for a local orthonormal frame $\{e_1, e_2\}$ of $T\Sigma$. This becomes $|\partial^\pi w|^2$ for a contact Cauchy-Riemann map w since $d^\pi w = \partial^\pi w + \bar{\partial}^\pi w$.

We now compute the Hodge Laplacian of e^π ,

$$\begin{aligned}
\frac{1}{2} \Delta e^\pi &= \frac{1}{2} \Delta |\partial^\pi w|^2 \\
&= -|\nabla^\pi(\partial^\pi w)|^2 - \langle \mathbf{tr}(\nabla^\pi)^2 \partial^\pi w, \partial^\pi w \rangle \\
&= -|\nabla^\pi(\partial^\pi w)|^2 + \langle (\nabla^\pi)^* \nabla^\pi \partial^\pi w, \partial^\pi w \rangle.
\end{aligned}$$

Using the Weitzenböck formula (4.2), we have derived the following general formula

$$\begin{aligned}
\frac{1}{2} \Delta e^\pi &= \frac{1}{2} \Delta |\partial^\pi w|^2 = -|\nabla^\pi(\partial^\pi w)|^2 + \langle \nabla^\pi)^* \nabla^\pi(\partial^\pi w), \partial^\pi w \rangle \\
&= -|\nabla^\pi(\partial^\pi w)|^2 + \langle \Delta^\pi \partial^\pi w, \partial^\pi w \rangle - \langle K^\pi(dw, dw) \partial^\pi w, \partial^\pi w \rangle - R\langle dw, \partial^\pi w \rangle \\
&= -|\nabla^\pi(\partial^\pi w)|^2 + \langle \Delta^\pi \partial^\pi w, \partial^\pi w \rangle - \langle K^\pi(dw, dw) \partial^\pi w, \partial^\pi w \rangle - R|\partial^\pi w|^2
\end{aligned}$$

where the last equality holds since $\partial^\pi w$ and $\bar{\partial}^\pi w$ are orthogonal to each other. Finally, applying Lemma 4.7, we have derived the following important formula for the Hodge Laplacian Δe^π

Theorem 4.8. *Let w be a contact Cauchy-Riemann map. Then*

$$\begin{aligned}
\frac{1}{2}\Delta e^\pi &= \frac{1}{2}\Delta|\partial^\pi w|^2 = -|\nabla^\pi(\partial^\pi w)|^2 + \langle(\nabla^\pi)^*\nabla^\pi(\partial^\pi w), \partial^\pi w\rangle \\
&= -|\nabla^\pi(\partial^\pi w)|^2 + \langle\Delta^\pi\partial^\pi w, \partial^\pi w\rangle - \langle K^\pi(dw, dw)\partial^\pi w, \partial^\pi w\rangle - R\langle dw, \partial^\pi w\rangle \\
&= -|\nabla^\pi(\partial^\pi w)|^2 + 2|\delta^{\nabla^\pi}\partial^\pi w|^2 - 2\delta\langle *d^{\nabla^\pi}\partial^\pi w, *\partial^\pi w\rangle \\
&\quad - \langle K^\pi(dw, dw)\partial^\pi w, \partial^\pi w\rangle - R|\partial^\pi w|^2.
\end{aligned} \tag{4.13}$$

5. $W^{k,2}$ -COERCIVE ESTIMATES: THE CLOSED CASE

In this section, we first derive the $W^{2,2}$ -coercive estimates for the equation (1.3) on the closed Riemann surface Σ , following the spirit of [Oh1, Chapter 7], which is the contact analog to [Oh1, Proposition 7.33]. Then by further taking higher derivatives, we derive general $W^{k,2}$ estimates.

5.1. $W^{2,2}$ estimates. Using the formula of Δe^π , we derive the following $W^{2,2}$ a priori estimates of the map w satisfying (1.3).

Theorem 5.1. *Assume Σ is a closed Riemann surface with finite number of marked points. Let $w : \dot{\Sigma} \rightarrow Q$ be any smooth solution to (1.3). Then*

$$\begin{aligned}
\int_{\Sigma} |\nabla dw|^2 &\leq \frac{1}{2}(C_1^2 + 1) \cdot \int_{\Sigma} |\partial^\pi w|^4 + \frac{3}{2}C_1^2 \int_{\Sigma} |w^*\lambda|^2 |\partial^\pi w|^2 \\
&\quad - 2 \min R \cdot \int_{\text{supp } R} |\partial^\pi w|^2
\end{aligned} \tag{5.1}$$

$$+ 2 \max |K^\pi| \int_{\Sigma} (|w^*\lambda|^2 |\partial^\pi w|^2 + |\partial^\pi w|^4) \tag{5.2}$$

where C_1 is the constant given by

$$C_1 = \|\mathcal{L}_{X_\lambda} J\|_{C^0}.$$

Proof. We start with (4.13)

$$\begin{aligned}
\frac{1}{2}\Delta e^\pi &= -|\nabla^\pi(\partial^\pi w)|^2 - 2\delta\langle *d^{\nabla^\pi}\partial^\pi w, *\partial^\pi w\rangle + 2|\delta^{\nabla^\pi}\partial^\pi w|^2 \\
&\quad - \langle K^\pi(dw, dw)\partial^\pi w, \partial^\pi w\rangle - R|\partial^\pi w|^2.
\end{aligned}$$

computed before in Theorem 4.8. We re-write this into

$$\begin{aligned}
|\nabla^\pi(\partial^\pi w)|^2 &= -\frac{1}{2}\Delta e^\pi - 2\delta\langle *d^{\nabla^\pi}\partial^\pi w, *\partial^\pi w\rangle + 2|\delta^{\nabla^\pi}\partial^\pi w|^2 \\
&\quad - \langle K^\pi(dw, dw)\partial^\pi w, \partial^\pi w\rangle - R|\partial^\pi w|^2.
\end{aligned} \tag{5.3}$$

On the other hand, using Gårding's identity (4.1), we obtain

$$|\nabla w^*\lambda|^2 = |dw^*\lambda|^2 + |\delta w^*\lambda|^2 = |dw^*\lambda|^2 = \frac{1}{4}|\partial^\pi w|^4 \tag{5.4}$$

for which we use $d(w^*\lambda \circ j) = 0$ for the second equality and Proposition 4.1 (2) for the last.

By adding up the two, we obtain

$$\begin{aligned}
&|\nabla^\pi(\partial^\pi w)|^2 + |\nabla(w^*\lambda)|^2 \\
&= -\frac{1}{2}\Delta e^\pi - 2\delta\langle *d^{\nabla^\pi}\partial^\pi w, *\partial^\pi w\rangle + 2|\delta^{\nabla^\pi}\partial^\pi w|^2 \\
&\quad - \langle K^\pi(dw, dw)\partial^\pi w, \partial^\pi w\rangle - R|\partial^\pi w|^2 + \frac{1}{4}|\partial^\pi w|^4.
\end{aligned} \tag{5.5}$$

Substituting the formula in Theorem 4.3, we derive

$$d^{\nabla^\pi} \partial^\pi w = -(w^* \lambda \circ j) \wedge \left(\frac{1}{2} (\mathcal{L}_{X_\lambda} J) \partial^\pi w \right).$$

Finally using (4.12), we obtain

$$\begin{aligned} |\delta^{\nabla^\pi} \partial^\pi w|^2 &= |d^{\nabla^\pi} \partial^\pi w|^2 \\ &= \left| w^* \lambda \circ j \wedge \left(\frac{1}{2} (\mathcal{L}_{X_\lambda} J) \partial^\pi w \right) \right|^2 \leq \frac{C_1^2}{4} |w^* \lambda|^2 |\partial^\pi w|^2 \end{aligned} \quad (5.6)$$

where

$$C_1 = \|\mathcal{L}_{X_\lambda} J\|_{C^0}.$$

We then note that

$$\begin{aligned} |\nabla dw|^2 &= |\nabla(\partial^\pi w) + \nabla(w^* \lambda X_\lambda)|^2 \\ &= 2|\nabla(\partial^\pi w)|^2 + 2|\nabla(w^* \lambda X_\lambda)|^2 \\ &= 2|\nabla(\partial^\pi w)|^2 + 2|\nabla(w^* \lambda) X_\lambda|^2 + 2|w^* \lambda|^2 |\nabla_{d^\pi w} X_\lambda|^2 \\ &= 2|\nabla(\partial^\pi w)|^2 + 2|\nabla(w^* \lambda)|^2 + 2|w^* \lambda|^2 |\nabla X_\lambda|^2 |d^\pi w|^2 \\ &\leq 2|\nabla(\partial^\pi w)|^2 + 2|\nabla(w^* \lambda)|^2 + \frac{1}{2} C_1^2 |w^* \lambda|^2 |d^\pi w|^2 \end{aligned} \quad (5.7)$$

$$= 2(|\nabla(\partial^\pi w)|^2 + |\nabla(w^* \lambda)|^2) + \frac{1}{2} C_1^2 |w^* \lambda|^2 |\partial^\pi w|^2 \quad (5.8)$$

where we use $\nabla_{X_\lambda} X_\lambda = 0$ and $\langle X_\lambda, \nabla X_\lambda \rangle = 0$ (and also $\langle \nabla^\pi X_\lambda, \nabla X_\lambda \rangle = 0$) for the third equality and

$$\nabla_{d^\pi w} X_\lambda = \frac{1}{2} (\mathcal{L}_{X_\lambda} J) J d^\pi w \quad (5.9)$$

for the second to the last. And we have the decomposition

$$\begin{aligned} |\nabla(\partial^\pi w)|^2 &= |\nabla^\pi(\partial^\pi w) + \langle X_\lambda, \nabla(\partial^\pi w) \rangle X_\lambda|^2 \\ &= |\nabla^\pi(\partial^\pi w)|^2 + |\langle X_\lambda, \nabla(\partial^\pi w) \rangle|^2 \\ &= |\nabla^\pi(\partial^\pi w)|^2 + |\langle \nabla_{d^\pi w} X_\lambda, \partial^\pi w \rangle|^2 \\ &\leq |\nabla^\pi(\partial^\pi w)|^2 + \frac{1}{4} C_1^2 |\partial^\pi w|^4 \end{aligned} \quad (5.10)$$

where we use $\bar{\partial}^\pi w = 0$ and $\langle \partial^\pi w, X_\lambda \rangle = 0$ for the second equality. Substituting this into (5.7), we obtain

$$\begin{aligned} |\nabla dw|^2 &\leq 2|\nabla^\pi(\partial^\pi w)|^2 + \frac{1}{2} C_1^2 |\partial^\pi w|^4 + 2|\nabla(w^* \lambda)|^2 + \frac{1}{2} C_1^2 |w^* \lambda|^2 |\partial^\pi w|^2 \\ &= 2(|\nabla^\pi(\partial^\pi w)|^2 + |\nabla(w^* \lambda)|^2) + \frac{1}{2} C_1^2 |\partial^\pi w|^4 + \frac{1}{2} C_1^2 |w^* \lambda|^2 |\partial^\pi w|^2. \end{aligned}$$

Now substituting (5.5) and (5.6) into this, we derive

$$\begin{aligned} |\nabla dw|^2 &\leq -\Delta e^\pi - 4\delta \langle *d^{\nabla^\pi} \partial^\pi w, * \partial^\pi w \rangle \\ &\quad - 2\langle K^\pi(dw, dw) \partial^\pi w, \partial^\pi w \rangle - 2R |\partial^\pi w|^2 \\ &\quad + \frac{1}{2} (C_1^2 + 1) |\partial^\pi w|^4 + \frac{3}{2} C_1^2 |w^* \lambda|^2 |\partial^\pi w|^2. \end{aligned} \quad (5.11)$$

Integrating (5.11) over Σ , we have finished the proof of Theorem 5.1. \square

An immediate corollary of this theorem, when combined with the standard Hölder's inequality and interpolation inequality between L^p -norms, is the following $W^{2,2}$ -coercive estimates

Corollary 5.2. *Let $\dot{\Sigma}$ and w be as above. Suppose w satisfies (1.3) on Σ and $|dw| \in L^2 \cap L^4$ on Σ . Then there exist uniform constants C_3, C_4 depending only on $\|K^\pi\|_{C^0}$, $\|R\|_{C^0}$ and $\|\mathcal{L}_{X_\lambda} J\|_{C^0}$ but independent of w such that*

$$\|dw\|_{W^{1,2}}^2 \leq C_3 \|dw\|_{L^4}^4 + C_4 \|dw\|_{L^2}^2.$$

The last statement follows from the standard bootstrapping argument by differentiating the equation (1.3).

We also need to have the following local version of the main estimates which is an important ingredient for the local regularity and the bubbling-off analysis.

Theorem 5.3. *Let $D = D^2(1)$ be the unit disc. There exists $C_5, C_6 > 0$ depending only on $D' \subset \overline{D}' \subset D$ and on $\|K^\pi\|_{C^0}$, $\|\mathcal{L}_{X_\lambda} J\|_{C^0}$ and $\|R\|_{C^0;D}$ but independent of w such that*

$$\|dw\|_{1,2,D'}^2 \leq C_5 \|dw\|_{4,D}^4 + C_6 \|dw\|_{2,D}^2$$

for any smooth map $w : D \rightarrow Q$ satisfying

$$\overline{\partial}^\pi w = 0, \quad d(w^* \lambda \circ j) = 0.$$

Proof. The proof is a standard practice in geometric analysis by multiplying a cut-off function ρ to dw such that $\rho \equiv 1$ on D' while $\rho \equiv 0$ outside $D \subset \Sigma$. We refer to [SU] for details in the context of harmonic maps and omit the details. \square

5.2. $W^{k,2}$ estimates for $k \geq 3$. Starting from the above $W^{2,2}$ -estimate, we proceed the higher $W^{k,2}$ -estimate inductively. For this purpose, consider the decomposition

$$dw = d^\pi w + w^* \lambda X_\lambda$$

and estimate $|\nabla^{k+1} d^\pi w|$ and $|\nabla^k (w^* \lambda X_\lambda)|$ inductively by alternatively bootstrapping starting from $k = 0$ as for the case of $|\nabla dw|$ in the previous subsection.

We start with estimating

$$\frac{1}{2} \Delta |(\nabla^\pi)^k d^\pi w|^2 = -|\nabla^\pi ((\nabla^\pi)^k d^\pi w)|^2 + \langle (\nabla^\pi)^* \nabla^\pi ((\nabla^\pi)^k d^\pi w), \nabla^k (d^\pi w) \rangle$$

similarly as for $\frac{1}{2} \Delta e^\pi = \frac{1}{2} \Delta |d^\pi w|^2$. Rewriting this and then combining the Weitzenböck formula applied to $(\nabla^\pi)^k d^\pi w$, we obtain

$$\begin{aligned} |\nabla^\pi ((\nabla^\pi)^k d^\pi w)|^2 &= -\frac{1}{2} \Delta |(\nabla^\pi)^k d^\pi w|^2 + \langle \Delta^\pi ((\nabla^\pi)^k d^\pi w), (\nabla^\pi)^k d^\pi w \rangle \\ &\quad - \langle K^\pi(dw, dw)(\nabla^\pi)^k d^\pi w, (\nabla^\pi)^k d^\pi w \rangle \\ &\quad - \langle R(\nabla^\pi)^k d^\pi w, (\nabla^\pi)^k d^\pi w \rangle. \end{aligned} \tag{5.12}$$

Therefore we have derived

$$\begin{aligned} \int_\Sigma |\nabla^\pi ((\nabla^\pi)^k d^\pi w)|^2 &= \int_\Sigma \langle \Delta^\pi ((\nabla^\pi)^k d^\pi w), (\nabla^\pi)^k d^\pi w \rangle \\ &\quad - \int_\Sigma \langle K^\pi(dw, dw)(\nabla^\pi)^k d^\pi w, (\nabla^\pi)^k d^\pi w \rangle \\ &\quad - \int_\Sigma \langle R(\nabla^\pi)^k d^\pi w, (\nabla^\pi)^k d^\pi w \rangle. \end{aligned}$$

Obviously the last two terms are bounded by the norm $\|dw\|_{k,2}^2$. It remains to examine the integral

$$\int_{\Sigma} \langle \Delta^{\pi}((\nabla^{\pi})^k d^{\pi} w), (\nabla^{\pi})^k d^{\pi} w \rangle.$$

Recalling $\Delta^{\pi} = d^{\nabla^{\pi}} \delta^{\nabla^{\pi}} + \delta^{\nabla^{\pi}} d^{\nabla^{\pi}}$, we rewrite

$$\int_{\Sigma} \langle \Delta^{\pi}((\nabla^{\pi})^k d^{\pi} w), (\nabla^{\pi})^k d^{\pi} w \rangle = \int_{\Sigma} |d^{\nabla^{\pi}}((\nabla^{\pi})^k d^{\pi} w)|^2 + \int_{\Sigma} |\delta^{\nabla^{\pi}}((\nabla^{\pi})^k d^{\pi} w)|^2.$$

On the other hand, we compute

$$d^{\nabla^{\pi}}((\nabla^{\pi})^k d^{\pi} w). \quad (5.13)$$

For this purpose, we quote the following lemma

Lemma 5.4. *For any ξ -valued one-form α ,*

$$d^{\nabla^{\pi}}(\nabla^{\pi} \alpha) = \nabla^{\pi}(d^{\nabla^{\pi}} \alpha) + w^* K^{\pi} \alpha \quad (5.14)$$

or equivalently

$$[d^{\nabla^{\pi}}, \nabla^{\pi}] \alpha = w^* K^{\pi} \alpha \quad (5.15)$$

for the commuator $[\cdot, \cdot]$.

Applying this to $d^{\nabla^{\pi}}((\nabla^{\pi})^k d^{\pi} w)$ iteratively, we derive

$$|d^{\nabla^{\pi}}((\nabla^{\pi})^k d^{\pi} w)|^2 \leq |(\nabla^{\pi})^k(d^{\nabla^{\pi}} d^{\pi} w)|^2 + G_k(|d^{\pi} w|, |w^* \lambda|) \quad (5.16)$$

where G_k is a polynomial function of $|d^{\pi} w|$, $|w^* \lambda|$ and their covariant derivatives upto order k . And applying the fundamental equation (4.6) to $d^{\nabla^{\pi}} d^{\pi} w$, the term itself has the bound

$$|(\nabla^{\pi})^k(d^{\nabla^{\pi}} d^{\pi} w)|^2 \leq H_k(|d^{\pi} w|, |w^* \lambda|)$$

for a polynomial function H_k of the form G_k .

Similarly, we obtain

$$|\delta^{\nabla^{\pi}}((\nabla^{\pi})^k d^{\pi} w)|^2 \leq I_k(|d^{\pi} w|, |w^* \lambda|)$$

for similar polynomial function I_k since $|\delta^{\nabla^{\pi}}((\nabla^{\pi})^k d^{\pi} w)|^2 = |d^{\nabla^{\pi}}((\nabla^{\pi})^k d^{\pi} w)|^2$.

We now summarize the above computations into

Proposition 5.5. *Let $w : \Sigma \rightarrow Q$ satisfy $\bar{\partial}^{\pi} w = 0$, $d(w^* \lambda \circ j) = 0$, on Σ . Then if $|d^{\pi} w| \in L^2 \cap L^4$ on $\dot{\Sigma}$,*

$$\int_{\dot{\Sigma}} |(\nabla^{\pi})^{k+1}(d^{\pi} w)|^2 \leq \int_{\dot{\Sigma}} J_k(|\partial^{\pi} w|, |w^* \lambda|) \quad (5.17)$$

for a polynomial function J_k of $|d^{\pi} w|$, $|w^* \lambda|$ its covariant derivatives up to 0, \dots, k of degree at most $2k + 4$.

Next we compute

$$\nabla^{k+1}(w^* \lambda X_{\lambda}) = \nabla^{k+1}(w^* \lambda) X_{\lambda} + \sum_{l=1}^{k+1} \binom{k+1}{l} \nabla^l(w^* \lambda) \nabla^{k+1-l} X_{\lambda}$$

Here we recall the formula ∇X_{λ} in (5.9). Therefore it follows that

$$\left| \sum_{l=1}^{k+1} \nabla^l(w^* \lambda) \nabla^{k+1-l} X_{\lambda} \right| \leq L_k(|d^{\pi} w|, |w^* \lambda|)$$

for a polynomial function L_k similar to J_k . We write

$$\begin{aligned} |\nabla^{k+1}(w^*\lambda)|^2 &= |\nabla(\nabla^k(w^*\lambda))|^2 \\ &= -\frac{1}{2}\Delta|\nabla^k(w^*\lambda)|^2 + \langle \nabla^*\nabla((\nabla^k(w^*\lambda))), \nabla^k(w^*\lambda) \rangle. \end{aligned}$$

Applying the Weitzenböck formula, we obtain

$$\nabla^*\nabla((\nabla^k(w^*\lambda))) = -\Delta(\nabla^k(w^*\lambda)) - R\nabla^k(w^*\lambda).$$

Therefore we have obtained

$$\begin{aligned} &\int_{\Sigma} \langle \nabla^*\nabla((\nabla^k(w^*\lambda))), \nabla^k(w^*\lambda) \rangle \\ &= -\int_{\Sigma} |d(\nabla^k(w^*\lambda))|^2 + |\delta(\nabla^k(w^*\lambda))|^2 - \int_{\Sigma} R|\nabla^k(w^*\lambda)|^2. \end{aligned}$$

By applying similar arguments considering the commutators $[d, \nabla^k]$, $[\delta, \nabla^k]$ and the equations $dw^*\lambda = \frac{1}{2}|d^\pi w|^2 dA$ from Proposition 4.1 and $\delta w^*\lambda = 0$, we have derived

Proposition 5.6. *Let $w : \Sigma \rightarrow Q$ satisfy $\bar{\partial}^\pi w = 0$, $d(w^*\lambda \circ j) = 0$. Then if $|d^\pi w| \in L^2 \cap L^4$,*

$$\int_{\Sigma} |\nabla^{k+1}w^*\lambda|^2 \leq L_k(|\partial^\pi w|, |w^*\lambda|)$$

for a polynomial function L_k of $|d^\pi w|$, $|w^\lambda|$ its covariant derivatives up to $0, \dots, k$ of degree at most $2k+3$.*

Now combining Propositions 5.5, 5.6, we derive

Theorem 5.7. *Let (Σ, j) be a closed Riemann surface. Let $w : \Sigma \rightarrow Q$ satisfy $\bar{\partial}^\pi w = 0$, $d(w^*\lambda \circ j) = 0$, on $\dot{\Sigma}$. Then if $|d^\pi w| \in L^2 \cap L^4$ and $|w^*\lambda| \in C^0$ on $\dot{\Sigma}$,*

$$\int_{\Sigma} |(\nabla)^{k+1}(dw)|^2 \leq \int_{\Sigma} J'_k(|\partial^\pi w|, |w^*\lambda|). \quad (5.18)$$

Here J'_{k+1} a polynomial function of covariant derivatives of $|d^\pi w|$, $|w^\lambda|$ up to $0, \dots, k$ with degree at most $2k+4$ whose coefficients are bounded by*

$$\|R\|_{C^k; \text{supp } R}, \|K^\pi\|_{C^k}, \|\mathcal{L}_{X_\lambda} J\|_{C^k}.$$

In particular,

$$\|dw\|_{k+1,2} \leq C_k(\|dw\|_{L^2}, \|dw\|_{L^4}) \quad (5.19)$$

for a similar polynomial function $C_k = C_k(s, t)$.

Proof. It remains to check the second statement, which itself follows expressing the bound of $\|dw\|_{k,2}^2$ inductively starting from $k=1$, i.e., Corollary 5.2. This finishes the proof. \square

Similar inductive computation also leads to the following local higher regularity.

Theorem 5.8. *Let $D = D^2(1)$ be the unit disc. There exists $C_{5;k}, C_{6;k} > 0$ depending only on $D' \subset \overline{D}' \subset D$ and on $\|K^\pi\|_{C^k}, \|\mathcal{L}_{X_\lambda} J\|_{C^k}$ and $\|R\|_{C^k; D}$ but independent of w such that for any smooth map $w : D \rightarrow Q$ satisfying*

$$\bar{\partial}^\pi w = 0, \quad d(w^*\lambda \circ j) = 0,$$

$$\|dw\|_{k+1,2; D'} \leq C_{k; D, D'}(\|dw\|_{2; D}, \|dw\|_{4; D})$$

for a similar polynomial function $C_{k; D, D'}(s, t)$ of s, t as C_k above depending also on D', D .

In particular, any weak solution of (1.3) in $W^{1,4}$ automatically lies in $W^{2,2}$ and becomes the classical solution to (1.3) and so becomes smooth.

6. ASYMPTOTIC CONVERGENCE TO REEB ORBITS AT PUNCTURE

We introduce the relevant energy in the asymptotic study of contact instanton map near the punctures of $\dot{\Sigma}$.

Let (Σ, j) be a compact Riemann surface and let $\{r_1, \dots, r_k\} \subset \Sigma$ be given marked points (including $k = 0$). Denote by $\dot{\Sigma}$ the associated punctured Riemann surface equipped with a Kähler metric h that is cylindrical on disjoint punctured discs $D_i \setminus \{0\}$ near each punctures r_i . We fix an associated isothermal coordinates $z_i = e^{-2\pi(\tau+it)}$ on each $D_i \setminus \{0\} \cong [0, \infty) \times S^1$.

The following is the relevant off-shell energy that controls the asymptotic behavior of the map near the punctures.

Definition 6.1. Let $w : \dot{\Sigma} \rightarrow Q$ be any smooth map and let r be a given puncture with isothermal coordinates z centered at r . We define

$$E_{(\lambda, J; D \setminus \{0\})}(w) = \frac{1}{2} \int_{[0, \infty) \times S^1} |d^\pi w|^2. \quad (6.1)$$

We put the following hypotheses in our asymptotic study of the finite energy contact instanton maps w :

Hypothesis 6.2. Let $r \in \{r_1, \dots, r_k\}$ be one of the given marked points and consider the map w restricted to the associated punctured disc $D \setminus \{r\}$.

- (1) Assume $w : [0, +\infty) \rightarrow Q$ is a contact instanton map, i.e., satisfies (1.3).
- (2) $E_{(\lambda, J; D \setminus \{0\})}(w) < \infty$,
- (3) $|\nabla w| < C$.
- (4) $\frac{1}{2} \int_{[0, \infty) \times S^1} |d^\pi w|^2 + \int_{S^1} w(0, \cdot)^* \lambda \neq 0$.

The following asymptotic convergence result of the finite energy contact instanton maps will be the fundamental ingredient for the applications thereof to the contact topology. The result essentially relies on our (local) $W^{3,2}$ a priori estimates which was already established in Theorem 5.8, and the finiteness of the above introduced end energy. This is an analog to Theorem 31 [H1] and its proof somewhat resembles that of [H1], [HWZ1, HWZ2] with the usage of bounded harmonic functions therein replaced by that of bounded harmonic one-forms. However, *we would like to emphasize that our proof does not involve symplectization*.

We denote

$$w^* \lambda = a_1 d\tau + a_2 dt, \quad \text{i.e., } a_1 = \lambda \left(\frac{\partial w}{\partial \tau} \right) \quad a_2 = \lambda \left(\frac{\partial w}{\partial t} \right).$$

Then

$$w^* \lambda \circ j = a_2 d\tau - a_1 dt.$$

The equation

$$d(w^* \lambda \circ j) = 0$$

from (1.3) implies that the divergence

$$\nabla \cdot w^* \lambda = \frac{\partial a_1}{\partial \tau} + \frac{\partial a_2}{\partial t} = 0.$$

Let w be as in Hypothesis 6.2. Then we can associate two natural asymptotic invariants of w defined by

$$T := \int_{[0,\infty) \times S^1} |d^\pi w|^2 + \int_{S^1} w(0, \cdot)^* \lambda \neq 0 \quad (6.2)$$

$$a := - \int_{S^1} w(0, \cdot)^* \lambda \circ j = \int_{S^1} \lambda \left(\frac{\partial w}{\partial \tau}(0, t) \right) dt. \quad (6.3)$$

Due to the closedness of $w(0, \cdot)^* \lambda \circ j$ the integral

$$\int_{S^1} w(\tau, \cdot)^* \lambda \circ j = -a$$

for all $\tau \geq 0$. We call T the *asymptotic contact action* and a the *asymptotic contact charge* of the instanton w .

Theorem 6.3 (Subsequence convergence). *Let w be as in Hypothesis 6.2. Then for any sequence $\tau_k \rightarrow \infty$, there exists a subsequence, still denote by τ_k , and a (nondegenerate) Reeb orbit γ with period, the action T , and with charge a such that*

$$\lim_{k \rightarrow \infty} w(\tau_k + \tau, t) = \gamma(a\tau + Tt)$$

uniformly on compact set $[-K, K] \times S^1$ for any given $K \geq 0$.

Proof. For any sequence $\tau_k \rightarrow \infty$, we can always choose a subsequence, still denoted by τ_k for $k = 1, 2, \dots$, so that

$$\lim_{k \rightarrow \infty} \int_{[\frac{1}{2}\tau_k, \infty) \times S^1} |d^\pi w|^2 = 0$$

by Hypothesis (2). We define a sequence of translated maps

$$w_k(\tau, t) = w(\tau + \tau_k, t) : [-\tau_k, \infty) \times S^1 \rightarrow Q$$

which gives rise to

$$\lim_{k \rightarrow \infty} \int_{[-\frac{1}{2}\tau_k, \infty) \times S^1} |d^\pi w_k|^2 = 0$$

We also have $|\nabla w_k| < C$ from Hypothesis (3) because the translations preserves the norm on the cylindrical metric near the puncture, and each w_k satisfies Hypothesis (1).

Let $K > 0$ be any given number and consider $[-K, K] \times S^1$ and note that eventually $[-K, K] \subset [-\frac{1}{2}\tau_k, \infty)$. Then the bound $\|w_k\|_{W^{3,2}([-K, K] \times S^1)} < C_K$ follows from compactness of Q and Theorem 5.8 for $k = 1$. Using the compactness of the embedding $W^{3,2} \hookrightarrow C^1$ on $[-K, K] \times S^1$, we get a subsequence w_k and $w_{\infty;K} : [-K, K] \times S^1 \rightarrow Q$ such that $w_k \rightarrow w_{\infty;K}$ in C^1 topology.

By letting $K \rightarrow \infty$ and taking the diagonal sequence argument, we obtain a map $w_\infty : \mathbb{R} \times S^1 \rightarrow Q$ such that $w_k \rightarrow w_\infty$ in compact C^1 topology on $\mathbb{R} \times S^1 \rightarrow Q$. We have

$$\|w_\infty\|_{C^1([-K, K] \times S^1)} \leq \sup_k \|w_k\|_{C^1([-K, K] \times S^1)}$$

which is uniformly bounded in the meaning that the upper bound is independent of K . In particular, we get $\|\nabla w_\infty\|_{C^0(\mathbb{R} \times S^1)} \leq C$.

Furthermore C^1 convergence gives that $w_{\infty,K} : [-K, K] \rightarrow Q$ satisfies

$$\begin{aligned} \int_{[-K,K] \times S^1} |d^\pi w_{\infty,K}|^2 &= \lim_{k \rightarrow \infty} \int_{[-K,K] \times S^1} |d^\pi w_k|^2 \\ &\leq \lim_{k \rightarrow \infty} \int_{[-\tau_k, \infty) \times S^1} |d^\pi w_k|^2 \\ &= 0 \end{aligned} \quad (6.4)$$

Hence $d^\pi w_\infty = 0$. Also since w_∞ also satisfies Hypothesis (1), we have $w_\infty^* d\lambda = \frac{1}{2} |d^\pi w_\infty|^2 dA$, which in turn implies

$$w_\infty^* d\lambda = 0. \quad (6.5)$$

The equation (6.5) indicates that $w_\infty^* \lambda$ is closed, i.e., $d(w_\infty^* \lambda) = 0$. Hence together with

$$\delta w_\infty^* \lambda = *d(w^* \lambda \circ j) = 0,$$

we derive

$$\Delta w_\infty^* \lambda = 0,$$

i.e. $w_\infty^* \lambda$ is a harmonic one-form on $\mathbb{R} \times S^1$ (with respect to the standard flat metric). It is also bounded by Hypothesis (2) on $\mathbb{R} \times S^1$. Therefore it follows that

$$w_\infty^* \lambda \circ j = a_{2,\infty} d\tau - a_{1,\infty} dt$$

for some constants $a_{1,\infty}, a_{2,\infty}$. By the connectedness of $[0, \infty) \times S^1$, the image of w is contained in a single leaf of the Reeb foliation. Let $\gamma : \mathbb{R} \rightarrow Q$ be a parameterization of the leaf so that $\dot{\gamma} = X_\lambda(\gamma)$. This parameterization is unique modulo a time-shift.

These imply that

$$w_\infty(\tau, t) = \gamma(a_{1,\infty} \tau + a_{2,\infty} t + c)$$

for a Reeb orbit γ as a map a priori defined on the universal covering space $\mathbb{R} \times \mathbb{R}$ of $\mathbb{R} \times S^1$ on the right hand side, where c can be chosen arbitrarily. But from $w(0, t+1) = w(0, t+1)$, we also obtain $\gamma(b(t+1)) = \gamma(bt)$ and hence γ is a closed period of period b .

By the C^1 -convergence of $w(\tau_k, t) \rightarrow w_\infty(0, t)$ it follows

$$\int_{S^1} w_\infty(0, \cdot)^* \lambda = \lim_{k \rightarrow \infty} \int_{S^1} w(\tau_k, \cdot)^* \lambda \quad (6.6)$$

$$\int_{S^1} w_\infty(0, \cdot)^* \lambda \circ j = \lim_{k \rightarrow \infty} \int_{S^1} w(\tau_k, \cdot)^* \lambda \circ j = -a. \quad (6.7)$$

From (6.7), we have shown $a_{1,\infty} = a$. For (6.6), the identity $\frac{1}{2} |d^\pi w|^2 dA = d(w^* \lambda)$ and Stokes' formula provides

$$\int_{S^1} w(\tau_k, \cdot)^* \lambda = \int_{S^1} w(0, \cdot)^* \lambda + \int_{[0, \tau_k] \times S^1} d(w^* \lambda) = \int_{S^1} w(0, \cdot)^* \lambda + \frac{1}{2} \int_{[0, \tau_k] \times S^1} |d^\pi w|^2.$$

Then using finiteness hypothesis of the integral $\int_{[0,\infty) \times S^1} |d^\pi w|^2 < \infty$ and Fatou's lemma, we derive

$$\begin{aligned}
a_{2,\infty} &= \int_{S^1} w_\infty(0, \cdot)^* \lambda \\
&= \lim_{k \rightarrow \infty} \int_{S^1} w(\tau_k, \cdot)^* \lambda \\
&= \lim_{k \rightarrow \infty} \left(\int_{S^1} w(0, \cdot)^* \lambda + \frac{1}{2} \int_{[0, \tau_k] \times S^1} |d^\pi w|^2 \right) \\
&= \int_{S^1} w(0, \cdot)^* \lambda + \frac{1}{2} \int_{[0, \infty) \times S^1} |d^\pi w|^2 = T.
\end{aligned}$$

Therefore $w_\infty(0, \cdot)$ is a Reeb orbit of period $T \neq 0$ with its charge a . \square

By specializing to the case $a = 0$, we have derived the following

Corollary 6.4. *Let w be as in Hypothesis 6.2 and assume $a = 0$ under the non-degeneracy hypothesis on the relevant Reeb orbit. Then for any sequence $\tau_k \rightarrow \infty$, there exists a subsequence, still denote by τ_k , and a (nondegenerate) Reeb orbit γ with period, the action T such that*

$$\lim_{k \rightarrow \infty} w(\tau_k + \tau, \cdot) = \gamma(t)$$

uniformly on compact set $[-K, K] \times S^1$ for any given $K \geq 0$.

In section 11, we will improve this subsequence convergence result to the full exponential convergence result for the case $a = 0$. We hope to come back to the corresponding convergence result for $a \neq 0$ elsewhere.

Remark 6.5. This corollary includes Hofer's subsequence convergence result in the standard case of symplectization of contact manifold in [H1]. It also covers the case of Hofer's generalized equation

$$\begin{cases} \bar{\partial}^\pi w = 0 \\ w^* \lambda \circ j = da + \gamma \end{cases} \quad (6.8)$$

introduced in [H2] in which γ is a smooth harmonic one-form on the closed Riemann surface Σ pulled back to $\dot{\Sigma}$ via the natural inclusion map $\dot{\Sigma} \hookrightarrow \Sigma$: This is because by definition, the charge vanishes, i.e., we have

$$\int_\alpha w^* \lambda \circ j = \int_\alpha da + \gamma = 0$$

for any local loop α around the given puncture.

7. FUNDAMENTAL EQUATION IN CYLINDRICAL COORDINATES

Consider the punctured Riemann surface $(\dot{\Sigma}, j)$ and fix a puncture. Equip a punctured neighborhood thereof with a cylindrical metric and let (τ, t) be the associated isothermal coordinates. We recall the decomposition of the w^*TQ -valued one-form dw on Σ ,

$$dw = d^\pi w + w^* \lambda X_\lambda$$

associated the splitting $TQ = \mathbb{R}\{X_\lambda\} \oplus \xi$ and the vector bundle connection ∇^π of $\xi \rightarrow Q$ is defined by

$$\nabla_X^\pi Z := \pi(\nabla_X Z) \quad (7.1)$$

for any vector fields Z tangent to ξ and X on Q . Denote

$$\begin{aligned} \zeta &= \pi \frac{\partial w}{\partial \tau}, \quad \eta = \pi \frac{\partial w}{\partial t} \\ w^* \lambda \circ j &= a_2 d\tau - a_1 dt, \end{aligned} \quad (7.2)$$

then we have

$$d^\pi w = \partial^\pi w = \zeta d\tau + \eta dt,$$

and

$$\nabla \cdot w^* \lambda = \frac{\partial a_2}{\partial t} + \frac{\partial a_1}{\partial \tau} = 0. \quad (7.3)$$

from (1.3).

We apply (4.6) to the pair $(\frac{\partial}{\partial \tau}, \frac{\partial}{\partial t})$. Then the left hand side becomes

$$J\nabla_\tau^\pi \zeta - \nabla_t^\pi \zeta,$$

while the right hand side becomes

$$\begin{aligned} & -\frac{1}{2} \lambda \left(\frac{\partial w}{\partial \tau} \right) (\mathcal{L}_{X_\lambda} J) \zeta - \frac{1}{2} \lambda \left(\frac{\partial w}{\partial t} \right) (\mathcal{L}_{X_\lambda} J) J \zeta \\ &= -\frac{1}{2} a_1 (\mathcal{L}_{X_\lambda} J) \zeta - \frac{1}{2} a_2 (\mathcal{L}_{X_\lambda} J) J \zeta, \end{aligned}$$

where we use (7.2) to get the second line. This immediately gives rise to the following form of fundamental equation in cylindrical coordinates, which is nothing but the linearization of $\bar{\partial}^\pi w = 0$ in the direction of $\zeta = \pi \frac{\partial w}{\partial \tau}$.

Proposition 7.1 (Fundamental equation in cylindrical coordinates). *Let $\zeta = \pi \frac{\partial w}{\partial \tau}$ as a section of $\xi \rightarrow Q$. Then*

$$\nabla_\tau^\pi \zeta + J\nabla_t^\pi \zeta - \frac{1}{2} a_2 (\mathcal{L}_{X_\lambda} J) \zeta + \frac{1}{2} a_1 (\mathcal{L}_{X_\lambda} J) J \zeta = 0. \quad (7.4)$$

Now we define an \mathbb{R} -family of operators

$$A_\tau : C^\infty(w_\tau^* \xi) \rightarrow C^\infty(w_\tau^* \xi)$$

with w_τ being the loop defined by $w_\tau(t) := w(\tau, t)$ defined by

$$A_\tau(Y) = J\nabla_t^\pi Y - \frac{1}{2} a_2 (\mathcal{L}_{X_\lambda} J) Y + \frac{1}{2} a_1 (\mathcal{L}_{X_\lambda} J) J Y \quad (7.5)$$

for $Y \in C^\infty(w_\tau^* \xi)$. This family of operators will enter in the study of perturbation results of the eigenvalues of the asymptotic operators at z which is the linearization operator

$$d\Upsilon_{(T, \lambda)} : C^\infty(z^* \xi) \rightarrow C^\infty(z^* \xi); \quad Y \mapsto \frac{DY}{dt} - T DX_\lambda(z)(Y)$$

of the map $\Upsilon_{(T, \lambda)}$ derived in Section 3 along the limit orbit z . Here z is determined by

$$z(t) = \lim_{\tau \rightarrow \infty} w(\tau, t).$$

By identifying $C^\infty(z^*\xi)$ with $C^\infty(S^1, w_\tau^*\xi)$ by suitable parallel transport for sufficiently large τ , it follows that the operator A_τ converges to $d\Upsilon_{(\lambda, T)}$ as $\tau \rightarrow \infty$. We refer readers to the next section for the precise meaning of this convergence.

When λ is nondegenerate and $w^*\lambda \circ j$ is exact, we will see that $a_1 \rightarrow 0$ uniformly. However when $w^*\lambda \circ j \neq 0$, a_1 will converge to the constant determined by

$$\int_{S^1} (w(\tau, \cdot))^* \lambda \circ j$$

which does not depend on τ by the closedness condition $d(w^*\lambda \circ j) = 0$.

8. $W^{k,2}$ -COERCIVE ESTIMATES: THE PUNCTURED CASE

In this section, we derive the $W^{k,2}$ estimates on punctured Riemann surfaces $\dot{\Sigma}$ equipped with cylindrical metric near the punctures.

We recall

$$\begin{aligned} |\nabla dw|^2 &\leq -\Delta e^\pi - 4\delta \langle *d^{\nabla^\pi} \partial^\pi w, *\partial^\pi w \rangle \\ &\quad - 2\langle K^\pi(dw, dw) \partial^\pi w, \partial^\pi w \rangle - 2R|\partial^\pi w|^2 \\ &\quad + \frac{1}{2}(C_1^2 + 1)|\partial^\pi w|^4 + \frac{3}{2}C_1^2|w^*\lambda|^2|\partial^\pi w|^2. \end{aligned}$$

from (5.11). Unlike the close case, the first two terms of the right hand side will give rise to some ‘asymptotic boundary terms’ after integration by parts. Therefore we need to impose some asymptotic boundary condition at the punctures.

For this purpose, we re-write the two terms as

$$-\Delta e^\pi dA = d(*de^\pi) \quad (8.1)$$

$$-\delta \langle *d^{\nabla^\pi} \partial^\pi w, *\partial^\pi w \rangle dA = d \langle *d^{\nabla^\pi} \partial^\pi w, *\partial^\pi w \rangle. \quad (8.2)$$

We denote by $\Sigma(\rho)$ the Riemann surface obtained by excising the discs $|z| \leq \rho$ around the punctures for the given analytic coordinates $z = e^{-2\pi(\tau+it)}$ centered at the punctures where (τ, t) is the corresponding cylindrical coordinates. Then we have its boundary

$$\partial\Sigma(\rho) = \bigcup_{\ell=1}^k \partial_\ell \Sigma(\rho) \quad (8.3)$$

where $\partial_\ell \Sigma(\rho)$ is the component of $\Sigma(\rho)$ associated to the ℓ -th puncture r_ℓ equipped with the boundary orientation of $\Sigma(\rho)$. In terms of the cylindrical coordinates (τ, t) , $\frac{\partial}{\partial \tau}$ corresponds to the outward normal vector of $\partial\Sigma(\rho)$.

By Stokes’ formula, we obtain

$$\begin{aligned} \int_{\Sigma(\rho)} -\Delta e^\pi dA &= \int_{\partial\Sigma(\rho)} *de^\pi = \sum_{\ell=1}^k \int_{\partial_\ell \Sigma(\rho)} *de^\pi \\ &= \sum_{\ell=1}^k \int_{\partial_\ell \Sigma(\rho)} \frac{\partial e^\pi}{\partial \tau} dt \\ &= \sum_{\ell=1}^k \frac{\partial}{\partial \tau} \int_{\partial_\ell \Sigma(\rho)} e^\pi(\tau, t) dt \\ &= 2 \sum_{\ell=1}^k \frac{\partial}{\partial \tau} \int_{\partial_\ell \Sigma(\rho)} |\zeta|^2 dt. \end{aligned} \quad (8.4)$$

For the term $-\delta\langle *d^{\nabla^\pi}\partial^\pi w, *\partial^\pi w \rangle dA$, we need some digression. In cylindrical coordinates (τ, t) (or in any isothermal coordinates on Σ), we recall $\partial^\pi w = \zeta d\tau + \eta dt$. where

$$\zeta = \pi \left(\frac{\partial w}{\partial \tau} \right), \quad \eta = \pi \left(\frac{\partial w}{\partial t} \right).$$

A straightforward calculation leads to the following general formulae.

Lemma 8.1. *Let w be any smooth map. Then*

$$\begin{aligned} \delta^{\nabla^\pi} \partial^\pi w &= -(\nabla_\tau^\pi \zeta + \nabla_t^\pi \eta) \\ *d^{\nabla^\pi} \partial^\pi w &= \nabla_\tau^\pi \eta - \nabla_t^\pi \zeta. \end{aligned}$$

In particular when w satisfies $\bar{\partial}^\pi w = 0$, we have $\eta = J\zeta$ and so

$$\partial^\pi w = \zeta d\tau + J\zeta dt, \quad *\partial^\pi w = -J\zeta d\tau + \zeta dt. \quad (8.5)$$

Then

$$*d^{\nabla^\pi} \partial^\pi w = d^{\nabla^\pi} \partial^\pi w \left(\frac{\partial}{\partial \tau}, \frac{\partial}{\partial t} \right) = J(\nabla_\tau^\pi \zeta + J\nabla_t^\pi \zeta) = J\bar{D}\zeta,$$

where we define

$$\bar{D}\zeta := \nabla_\tau^\pi \zeta + J\nabla_t^\pi \zeta.$$

Then we obtain

$$\begin{aligned} \int_{\Sigma(\rho)} -\delta\langle *d^{\nabla^\pi} \partial^\pi w, *\partial^\pi w \rangle dA &= \int_{\Sigma(\rho)} d * \langle *d^{\nabla^\pi} \partial^\pi w, *\partial^\pi w \rangle \\ &= \sum_{\ell=1}^k \int_{\partial_\ell \Sigma(\rho)} * \langle *d^{\nabla^\pi} \partial^\pi w, *\partial^\pi w \rangle dt \\ &= \sum_{\ell=1}^k \int_{\partial_\ell \Sigma(\rho)} * \langle J\bar{D}\zeta, (-J\zeta d\tau - \zeta dt) \rangle \\ &= \sum_{\ell=1}^k \int_{\partial_\ell \Sigma(\rho)} -\langle \bar{D}\zeta, \zeta \rangle dt. \end{aligned}$$

Using the fundamental equation in the cylindrical coordinate derived in Proposition 7.1, we obtain

$$\bar{D}\zeta = \frac{1}{2}a_2(\mathcal{L}_{X_\lambda} J)\zeta - \frac{1}{2}a_1(\mathcal{L}_{X_\lambda} J)J\zeta \quad (8.6)$$

and hence

$$\int_{\partial_\ell \Sigma(\rho)} -\langle \bar{D}\zeta, \zeta \rangle dt = \int_{\partial_\ell \Sigma(\rho)} \left\langle \frac{1}{2}a_1(\mathcal{L}_{X_\lambda} J)J\zeta - \frac{1}{2}a_2(\mathcal{L}_{X_\lambda} J)\zeta, \zeta \right\rangle dt. \quad (8.7)$$

Motivated by these explicit formulae, we introduce the following function

$$f(\tau) = \frac{1}{2} \int_{S^1} e^\pi(\tau, t) dt = \int_{S^1} |\zeta|^2(\tau, t) dt \quad (8.8)$$

which will be also important for the later study of exponential convergence in part 2.

We then obtain

$$\begin{aligned}
\int_{\Sigma(\rho)} -\Delta e^\pi dA &= \sum_{\ell=1}^k f'_\ell(\tau) \\
\int_{\partial_\ell \Sigma(\rho)} |\langle \bar{D}\zeta, \zeta \rangle| dt &\leq C_1(\|a_1\|_{C^0} + \|a_2\|_{C^0}) \sum_{\ell=1}^k \int_{\partial_\ell \Sigma(\rho)} |\zeta|^2 dt \\
&\leq 2C_1 \|w^* \lambda\|_{C^0} \sum_{\ell=1}^k f_\ell(\tau)
\end{aligned} \tag{8.9}$$

from (8.4) and (8.7) respectively, where $C_1 = \|\mathcal{L}_{X_\lambda} J\|_{C^0}$, and f_ℓ is the L^2 -integral function f above defined on the puncture disc around r_ℓ .

The following is the $W^{2,2}$ a priori estimates on the punctured case which is the counterpart of Theorem 1.6 of the closed case. The proof is the same as that of Theorem 1.6 incorporating the asymptotic boundary condition.

Theorem 8.2. *Let (Σ, j) be a closed Riemann surface with a finite number of marked points $\{r_1, \dots, r_k\}$. Denote by $\dot{\Sigma}$ the associated punctured Riemann surface with a Kähler metric h on (Σ, j) which is cylindrical near the puncture. Let f_ℓ be the function defined as above associated to the ℓ -th puncture r_ℓ . Suppose w satisfies (1.3) on $\dot{\Sigma}$ and $|d^\pi w| \in L^2 \cap L^4$ and $|w^* \lambda| \in C^0$ on $\dot{\Sigma}$*

$$\begin{aligned}
\lim_{\tau \rightarrow \infty} a_1 &= a, & \lim_{\tau \rightarrow \infty} a_2 &= T \\
\lim_{\tau \rightarrow \infty} f_\ell(\tau) &= 0, & \lim_{\tau \rightarrow \infty} f'_\ell(\tau) &= 0
\end{aligned} \tag{8.10}$$

for all $\ell = 1, \dots, k$. Then

$$\begin{aligned}
\int_{\dot{\Sigma}} |\nabla dw|^2 &\leq \frac{1}{2}(C_1^2 + 1) \|\partial^\pi w\|_{L^4}^4 - 2 \min R \|\partial^\pi w\|_{L^2, \supp R}^2 + \frac{3}{2} C_1^2 \|w^* \lambda\|_{C^0}^2 \|\partial^\pi w\|_{L^4}^2 \\
&\quad + 2 \max |K^\pi| (\|w^* \lambda\|_{C^0}^2 \|\partial^\pi w\|_{L^2}^2 + \|\partial^\pi w\|_{L^4}^4).
\end{aligned}$$

Here K^π is the curvature of ∇^π , R the Gaussian curvature of the metric h on Σ and

$$C_1 = \|\mathcal{L}_{X_\lambda} J\|_{C^0}.$$

We would like to remark that the asymptotic boundary conditions imposed in this theorem will be automatically satisfied under the Hypothesis 6.2 together with nondegeneracy of the asymptotic Reeb orbits obtained from subsequence convergence theorem, Theorem 6.3. These will be established in part 2.

An immediate corollary of this theorem, when combined with the standard Hölder's inequality, is the following $W^{2,2}$ -coercive estimates

Corollary 8.3. *Let $\dot{\Sigma}$ and w be as above. Suppose w satisfies (1.3) on $\dot{\Sigma}$ and $|d^\pi w| \in L^2 \cap L^4$ and $|w^* \lambda| \in C^0$ on $\dot{\Sigma}$ and assume (8.10). Then there exists uniform constants C'_3, C'_4 depending only on*

$$\|K^\pi\|_{C^0}, \|R\|_{C^0}, \|\mathcal{L}_{X_\lambda} J\|_{C^0}, \|w^* \lambda\|_{C^0}$$

but independent of w such that

$$\|dw\|_{W^{1,2}}^2 \leq C'_3 \|\partial^\pi w\|_{L^4}^4 + C'_4 \|\partial^\pi w\|_{L^2}^2.$$

Once we establish the above $W^{2,2}$ -estimate, the alternating inductive bootstrap-arguments will establish the following higher $W^{k,2}$ -estimate.

Theorem 8.4. *Let (Σ, j) be a closed Riemann surface. Let $w : \Sigma \rightarrow Q$ satisfy $\bar{\partial}^\pi w = 0$, $d(w^* \lambda \circ j) = 0$, on $\dot{\Sigma}$ and (8.10). Then if $|d^\pi w| \in L^2 \cap L^4$ and $|w^* \lambda| \in C^0$ on $\dot{\Sigma}$,*

$$\int_{\dot{\Sigma}} |(\nabla)^{k+1}(dw)|^2 \leq \int_{\dot{\Sigma}} J'_k(|d^\pi w|, |w^* \lambda|). \quad (8.11)$$

Here J'_{k+1} a polynomial function of covariant derivatives of $|d^\pi w|$, $|w^* \lambda|$ up to $0, \dots, k$ with degree at most $2k + 4$ whose coefficients are bounded by

$$\|R\|_{C^k; \text{supp } R}, \|K^\pi\|_{C^k}, \|\mathcal{L}_{X_\lambda} J\|_{C^k}.$$

In particular,

$$\|dw\|_{k+1,2} \leq C_k(\|dw\|_{L^2}, \|dw\|_{L^4}) \quad (8.12)$$

for a similar polynomial function $C_k = C_k(s, t)$.

Remark 8.5. We would like to note that the starting from the asymptotic decay condition (??) of L^2 -integral of e^π over S^1 , we also need to inductively establish the corresponding decay for $W^{k,2}$ -integral of e^π over S^1 itself as a part of component in the alternating bootstrapping process. In summary, on the punctured Riemann surface $\dot{\Sigma}$, the proofs of the above $W^{k,2}$ -estimates and of the exponential estimates should be performed together simultaneously in the alternating bootstrapping process performed in here and in part (2).

9. APPENDIX: WEDGE PRODUCT OF VECTOR-VALUED FORMS

In this section, we briefly recall the definition of wedge product of vector bundle-valued forms and give the proof of two lemmas whose proofs are postponed.

We first recall the E -valued differential k -form α for a vector bundle $E \rightarrow M$ in general.

Definition 9.1. Let $E \rightarrow M$ be a vector bundle. A E -valued k -form is a section of the bundle $\Lambda^k(M) \otimes E$. We denote by $\Omega^k(E)$ the set of E -valued k -forms.

We will need to consider only the cases of zero, one and two forms and so restrict our discussion to those cases from now on.

Now suppose that E carries an inner product. Let α and β be E -valued 0-form and 1-form respectively. Then we define the inner product operation

$$(\alpha, \beta) \rightarrow \langle \alpha, \beta \rangle \in \Omega^1(M)$$

to be the one characterized by the equation

$$\langle \alpha, \beta \rangle(X) = \langle \alpha, \beta(X) \rangle \quad (9.1)$$

for any vector field X on M . Here we note that both α and $\beta(X)$ are sections of E and so the inner product is well-defined.

We now specialize to the case of $M = \Sigma$ with (Σ, j) being a Riemann surface and let h be an associated Kähler metric thereof. We denote by ∇^{LC} its Levi-Civita connection. We also assume that E carries a connection ∇ preserving the inner product on E . We define the *wedge product*, denoted by $\omega_1 \wedge \omega_2$, of two E -valued one-forms ω_1 and ω_2 . This is characterized by the equation

$$\omega_1 \wedge \omega_2(X, Y) = \langle \omega_1(X), \omega_2(Y) \rangle - \langle \omega_2(X), \omega_1(Y) \rangle \quad (9.2)$$

for any two vector fields X, Y on M .

We restate Lemma 4.4 here.

Lemma 9.2. *Assume α is a zero form in $\Omega^0(w^*\xi)$ and β is a one-form in $\Omega^1(w^*\xi)$. $\langle \cdot, \cdot \rangle$ is the inner production on $w^*\xi$ introduced from the metric of Q . Then we have*

$$\langle d^{\nabla^\pi} \alpha, \beta \rangle - \langle \alpha, \delta^{\nabla^\pi} \beta \rangle = -\delta \langle \alpha, \beta \rangle.$$

Proof. We compute

$$\begin{aligned} -\delta \langle \alpha, \beta \rangle &= *d^{\nabla^\pi} * \langle \alpha, \beta \rangle = *d^{\nabla^\pi} \langle \alpha, *\beta \rangle \\ &= *(d^{\nabla^\pi} \alpha \wedge *\beta) + *\langle \alpha, d^{\nabla^\pi} (*\beta) \rangle \\ &= *\langle d^{\nabla^\pi} \alpha, \beta \rangle d \text{vol} + \langle \alpha, *d^{\nabla^\pi} (*\beta) \rangle \\ &= \langle d^{\nabla^\pi} \alpha, \beta \rangle - \langle \alpha, \delta^{\nabla^\pi} \beta \rangle. \end{aligned}$$

In the third line, we also use the fact that our connection is a Riemannian connection and here one should extend the operation \wedge to the vector-forms in the way that the product is taking the inner product in the fiber direction and take the wedge product on the base. \square

We now restate Lemma 4.5 here.

Lemma 9.3. *For any connection ∇ and vector-valued one-form α ,*

$$|\nabla \alpha|^2 = |d^\nabla \alpha|^2 + |\delta^\nabla \alpha|^2 - *(\nabla \alpha \wedge \nabla_{j(\cdot)} \alpha).$$

Proof. Note that the equality is a pointwise equality. Let $z \in \Sigma$ be a given point. We choose an orthonormal local frame $\{e_1, e_2\}$ of $T\Sigma$ so that $e_2(z) = j_z e_1(z)$

$$(\nabla e_1)(z) = 0 = (\nabla e_2)(z).$$

If we denote its dual frame by $\{\theta^1, \theta^2\}$, then we have

$$(\nabla \theta^1)(z) = 0 = (\nabla \theta^2)(z).$$

We express

$$\alpha = \zeta \theta^1 + \eta \theta^2$$

for some (locally defined) sections of ξ near z . Then we have

$$(\nabla \alpha)(z) = (\nabla \zeta)(z) \theta^1(z) + (\nabla \eta)(z) (\theta^2)(z).$$

Similarly we obtain

$$(\nabla_{j(\cdot)} \alpha)(z) = (\nabla_{j(\cdot)} \zeta)(z) (\theta^1)(z) + (\nabla_{j(\cdot)} \eta)(z) \theta^2(z)$$

First we compute

$$\begin{aligned} d^\nabla \alpha &= (\nabla_{e_1} \eta - \nabla_{e_2} \zeta) \theta^1 \wedge \theta^2 \\ \delta^\nabla \alpha &= \nabla_{e_1} \zeta + \nabla_{e_2} \eta. \end{aligned}$$

Therefore we obtain

$$\begin{aligned} |d^\nabla \alpha|^2 + |\delta^\nabla \alpha|^2 &= |\nabla_{e_1} \eta|^2 + |\nabla_{e_2} \zeta|^2 + |\nabla_{e_1} \zeta|^2 + |\nabla_{e_2} \eta|^2 \\ &\quad + 2(\langle \nabla_{e_1} \zeta, \nabla_{e_2} \eta \rangle - \langle \nabla_{e_1} \eta, \nabla_{e_2} \zeta \rangle) \\ &= |\nabla \alpha|^2 + 2(\langle \nabla_{e_1} \zeta, \nabla_{e_2} \eta \rangle - \langle \nabla_{e_1} \eta, \nabla_{e_2} \zeta \rangle). \end{aligned}$$

It remains to prove

$$*(\nabla \alpha \wedge \nabla_{j(\cdot)} \alpha) = 2(\langle \nabla_{e_1} \zeta, \nabla_{e_2} \eta \rangle - \langle \nabla_{e_1} \eta, \nabla_{e_2} \zeta \rangle). \quad (9.3)$$

Taking the wedge product, we get the equality

$$\nabla \alpha \wedge \nabla_{j(\cdot)} \alpha = \langle \nabla \zeta, \nabla_{j(\cdot)} \eta \rangle \theta^1 \wedge \theta^2 - \langle \nabla_{j(\cdot)} \zeta, \nabla \eta \rangle \theta^1 \wedge \theta^2$$

at z . Therefore

$$*(\nabla\alpha \wedge \nabla_{j(\cdot)}\alpha) = \langle \nabla\zeta, \nabla_{j(\cdot)}\eta \rangle - \langle \nabla_{j(\cdot)}\zeta, \nabla\eta \rangle$$

at z . But we evaluate

$$\langle \nabla\zeta, \nabla_{j(\cdot)}\eta \rangle = \langle \nabla_{e_1}\zeta, \nabla_{e_2}\eta \rangle - \langle \nabla_{e_2}\zeta, \nabla_{e_1}\eta \rangle$$

at z . Similarly we obtain the equality

$$\langle \nabla_{j(\cdot)}\zeta, \nabla\eta \rangle = \langle \nabla_{e_2}\zeta, \nabla_{e_1}\eta \rangle - \langle \nabla_{e_1}\zeta, \nabla_{e_2}\eta \rangle$$

at z . Subtracting the latter from the first, we obtain

$$*(\nabla\alpha \wedge \nabla_{j(\cdot)}\alpha) = 2\langle \nabla_{e_1}\zeta, \nabla_{e_2}\eta \rangle - 2\langle \nabla_{e_2}\zeta, \nabla_{e_1}\eta \rangle$$

at z . Since the right hand side does not depend on

Since z is arbitrary, we have finished the proof of (9.3). This finishes the proof. \square

Part 2. C^∞ exponential estimates on cylindrical ends

In this part, we illustrate usefulness of the usage of this contact triad connection ∇ on Q and on the Hermitian vector bundle $\xi \rightarrow Q$ in tensorial calculations of the sections of ξ in cylindrical coordinates near the given puncture. More specifically we use this connection to differentiate $\zeta = \pi\left(\frac{\partial w}{\partial \tau}\right)$ and prove the exponential convergence property of the contact Cauchy-Riemann map w satisfying

$$\bar{\partial}^\pi w = 0, \quad d(w^*\lambda \circ j) = 0$$

to a Reeb orbit z as $\tau \rightarrow \infty$ when z is nondegenerate. The convergence is uniform when the uniform gradient bound

$$|\nabla w| < C$$

is assumed, which will be always achieved after bubbling-off analysis which is by now standard. Of course, this derivation covers the exact case, i.e, the case of pseudoholomorphic maps $(a, w) : \dot{\Sigma} \rightarrow \mathbb{R} \times Q$, for which $w^*\lambda \circ j = da$.

Some comparison between our proof of exponential convergence and that of Hofer-Wysocki-Zehnder [HWZ1, HWZ2] may be useful.

First of all, our proof covers the more general case with the equation $w^*\lambda \circ j = da$ replaced by the closed condition $d(w^*\lambda \circ j) = 0$. By doing so, we can completely get rid of symplectization picture out of our derivation.

In the case of [HWZ1, HWZ2], the authors used a normal form theorem of the neighborhood of a nondegenerate Reeb orbit and then represent the map w in some special coordinate as $w = (a, u) \in S^1 \times \mathbb{R}^2$ (in 3 dimension) and transfer J and others to $S^1 \times \mathbb{R}^2$. They first study the L^2 -exponential decay of the loops $t \mapsto u(\tau, t)$ as $\tau \rightarrow \infty$. Because of the complication of the push-forward J in coordinate representation, the computations involved are quite messy and requires some ingenuity in their logical arguments.

On the other hand, we directly study the derivative dw as a vector valued one-form on $\mathbb{R} \times S^1$ by exploiting the presence of splitting $TQ = \mathbb{R}\{X_\lambda\} \oplus \xi$ and so

$$dw = \partial^\pi w + w^*\lambda X_\lambda.$$

In cylindrical coordinates, we have

$$\partial^\pi w = \zeta d\tau + J\zeta dt, \quad w^*\lambda \circ j = a_2 d\tau - a_1 dt$$

where we denote

$$\pi \frac{\partial w}{\partial \tau} = \zeta, \quad \pi \frac{\partial w}{\partial t} = J\zeta, \quad a_1 = \lambda \left(\frac{\partial w}{\partial \tau} \right), \quad a_2 = \lambda \left(\frac{\partial w}{\partial t} \right).$$

As sections on the circle S^1 , we have

$$\begin{aligned} \frac{\partial w}{\partial \tau} &= \pi \frac{\partial w}{\partial \tau} + a_1 X_\lambda(w) \\ \frac{\partial w}{\partial t} &= \pi \frac{\partial w}{\partial t} + a_2 X_\lambda(w). \end{aligned}$$

We first perform the tensorial calculation of $\zeta = \pi \frac{\partial w}{\partial \tau}$ as a section of $w(\tau, \cdot)^* \xi$ in terms of the contact Hermitian connection ∇^π (and the canonical contact connection ∇ on Q). As a consequence, after this step, we obtain a stronger $W^{1,2}$ -exponential estimate for ζ , rather than just L^2 . (By the way, this ζ roughly corresponds to the first derivative of the map u appearing in [HWZ1, HWZ2].) Once we have proved this $W^{1,2}$ -exponential estimate, the C^0 -convergence of the map w to z and standard bootstrapping argument via the elliptic estimates immediately give rise to the exponential convergence in all higher orders.

Besides the above mentioned methodological differences, one major conceptual difference between our proof of C^∞ exponential convergence and that of [HWZ1, HWZ2] is that our proof is based on the purely inductive bootstrapping arguments, while the latter needed to use C^∞ asymptotic decay (or more precisely C^2 asymptotic decay) even for the proof of C^1 -exponential convergence. Our proof has been possible because of the usage of contact triad connection and the associated fundamental equation which precisely reflects the geometry behind the contact Cauchy-Riemann map so that they interact nicely to reveal such an inductive bootstrapping procedure. In that regard, we have been particularly keen to make sure that the final C^1 -exponential estimate and its proof does not depend on the estimates of the second (or higher) derivatives of the contact Cauchy-Riemann map w (or the function a associated to the pseudoholomorphic map (a, w) in the symplectization.) It turns out that this task involves highly non-trivial combinatorics of tensor calculations and depends on our choice of special connections and precise tensorial computations although the calculations themselves are quite pedestrian. It will be clear from the way how the needed estimates are carried out that without the usage of these special connections the current inductive alternating bootstrapping exponential estimates would not have been possible.

In this part, we add the following additional nondegeneracy hypothesis of the relevant Reeb orbits.

Hypothesis 9.4 (Nondegeneracy). Assume that the T -periodic orbit in Hypothesis 6.2 is nondegenerate: If $z = \gamma(T \cdot)$ for a nondegenerate T -periodic Reeb orbit γ on Q , we denote the S^1 -family of rotations of the loop $z : S^1 \rightarrow Q$ by

$$Z = \{z_\theta \in C^1(S^1, Q) \mid z_\theta(t) := z(t - \theta), \theta \in S^1\}.$$

10. ASYMPTOTIC PERTURBATION RESULTS OF EIGENVALUES

We recall from Section 3 that the linearization operator of the Reeb orbit z

$$A_z : W^{1,2}(z^* \xi) \rightarrow L^2(z^* \xi),$$

has the form

$$\begin{aligned} A_z(\eta) &= Jd(\Upsilon_T)_z(\eta) = J\frac{D\eta}{dt} - TJD X_\lambda(\eta) \\ &= J\frac{D\eta}{dt} - \frac{1}{2}T(\mathcal{L}_{X_\lambda}J)\eta. \end{aligned} \quad (10.1)$$

Nondegeneracy hypothesis of z implies $\ker A_z = \{0\}$ and then since the Fredholm index of A_z is zero its cokernel is also trivial. We note that the operator $A_z : L^2 \rightarrow L^2$ is a self-adjoint unbounded operator and so has real eigenvalues. It follows from the open mapping theorem that there exists $\delta = \delta(z) > 0$ such that

$$\|A_z\eta\|_{L^2} \geq \delta\|\eta\|_{L^2}$$

for all $\eta \in W^{1,2}(S^1, z^*\xi)$. Since the rotation $R_\theta : z \rightarrow z_\theta$ induces an isometry between the relevant Sobolev spaces of $W^{1,2}$, L^2 associated to z and z_θ , we have derived the following lemma

Lemma 10.1. *There exists $\lambda_1 > 0$ such that for any $z \in Z$,*

$$\|A_z\eta\|_{L^2} \geq \lambda_1\|\eta\|_{L^2} \quad (10.2)$$

for all $\eta \in W^{1,2}(z^*\xi)$.

We will also need the following technical lemma. (We need only the version with $W^{1,2}$ replaced by C^1 in this lemma but would like to give this stronger version since the proofs will not make much difference.)

Lemma 10.2. *Consider the exponential map*

$$\widetilde{\exp}_z : W^{1,2}(z^*TQ) \rightarrow W^{1,2}(S^1, Q)$$

defined by $\widetilde{\exp}_z(v)(t) = \exp_{z(t)}(v(t))$ for $v \in W^{1,2}(z^*TQ)$ and define the parameterized map

$$\widetilde{\exp}_Z : \bigcup_{z \in Z} W^{1,2}(z^*TQ) \rightarrow W^{1,2}(S^1, Q)$$

by $\widetilde{\exp}_Z(z, v) = \exp_z(v)$. For any given $\epsilon > 0$, consider the image

$$N_\epsilon(Z) := \bigcup_{z \in Z} \widetilde{\exp}_z(W^{1,2}(z, \epsilon)) \subset W^{1,2}(S^1, Q)$$

where

$$W^{1,2}(z, \epsilon) = \{v \in W^{1,2}(z^*TQ) \mid \|v\|_{1,2} \leq \epsilon\}.$$

Then the image is compact in C^δ -topology with $0 \leq \delta < \frac{1}{2}$.

Proof. This is an immediate consequence of compactness of Z and the compactness of the embedding $W^{1,2}(S^1, Q) \hookrightarrow C^\delta(S^1, Q)$. \square

Using this lemma and Theorem 6.3 and Hypothesis 6.2 (4), we now prove

Proposition 10.3. *Assume the contact charge vanishes, i.e., $a = 0$. Let $\lambda_1 > 0$ be the constant given in Lemma 10.1. Consider the completion of the operator (7.5)*

$$A_\tau : W^{1,2}(w(\tau, \cdot)^*\xi) \rightarrow L^2(w(\tau, \cdot)^*\xi).$$

Then there exists some $\tau_1 \in \mathbb{R}$ such that for all $\tau \geq \tau_1$,

$$\|A_\tau(\eta)\|_{L^2} \geq \frac{3}{4}\lambda_1\|\eta\|_{L^2} \quad (10.3)$$

for all $\eta \in W^{1,2}(w(\tau, \cdot)^*\xi)$.

Proof. First of all, Hypothesis 6.2 and Theorem 6.3 imply that for any given $\epsilon > 0$, there exists some $\tau_1 > 0$ such that

$$w(\tau, \cdot) \in N_\epsilon(Z) \quad (10.4)$$

for all $\tau \geq \tau_1$.

Again by Hypothesis 6.2 and Theorem 6.3, it follows that the union

$$\{w(\tau, \cdot)\}_{\tau \geq \tau_1} \cup Z$$

is a (fiberwise) closed bounded subset of $N_\epsilon(Z) \subset W^{1,2}(S^1, Q)$. Therefore it is compact in C^ϵ -topology. Now for each given τ , we consider $z_\tau \in Z$ that is the shortest distance point of Z from $w(\tau, \cdot)$, i.e.,

$$d_{C^\epsilon}(w(\tau, \cdot), z) = d_{C^\epsilon}(w(\tau, \cdot), Z) = \min_{z \in Z} d_{C^\epsilon}(w(\tau, \cdot), z).$$

Such z_τ exists by the compactness of the subset $Z \subset C^0$. There could be more than one such z_τ but any one choice will do our purpose. We denote by $\Pi = \Pi_{z_\tau}^{w(\tau, \cdot)}$ the parallel transport map of the vector bundle ξ with respect to ∇^π from z_τ to $w(\tau, \cdot)$ along the short geodesic between $z_\tau(t)$ and $w(\tau, t)$ at each $t \in S^1$. Then we consider the operator

$$\Pi^{-1} A_\tau \Pi : W^{1,2}(z_\tau^* \xi) \rightarrow W^{1,2}(z_\tau^* \xi)$$

The following lemma is the key ingredient in our optimal exponential decay estimate under the main hypothesis put in this beginning of the section, but without assuming any assumption on the higher derivatives of w . (The latter will be an automatic consequence by our bootstrap arguments.)

Lemma 10.4. *There exists some τ_2 such that for all $\tau \geq \tau_2$*

$$|(\Pi^{-1} A_\tau \Pi - A_{z_\tau})\eta| \leq \frac{1}{4} \lambda_1 |\eta|$$

for any $\eta \in C^1(z_\tau^* \xi)$.

Proof. We recall

$$A_\tau = J \nabla_t^\pi + B$$

where B is the zero-order operator on $w(\tau, \cdot)^* \xi$ by

$$B\eta = -\frac{1}{2} a_2(\mathcal{L}_{X_\lambda} J)\eta + \frac{1}{2} a_1(\mathcal{L}_{X_\lambda} J)J\eta \quad (10.5)$$

for $\eta \in w(\tau, \cdot)^* \xi$. Then using the J -linearity of the Hermitian connection ∇^π , we compute

$$\begin{aligned} (\Pi^{-1} A_\tau \Pi)\eta &= \Pi^{-1} (J \nabla_t^\pi + B)(\Pi\eta) \\ &= J \Pi^{-1} \nabla_t^\pi (\Pi\eta) + \Pi^{-1} B \Pi(\eta). \end{aligned} \quad (10.6)$$

Now for given τ , we consider the map

$$\Gamma(s, t) = \exp_{z_\tau(t)}(sE(z_\tau(t), w(\tau, t)))$$

where we recall the definition $E(x, y) = \exp_x^{-1} y$. Then by definition of the parallel transport, we have

$$\Pi\eta(t) = \Xi(1, t)$$

where $\Xi = \Xi(s, t)$ is the solution to the ordinary differential equation

$$\nabla_s^\pi \Xi = 0, \quad \Xi(0, t) = \eta(t)$$

which defines the parallel transport of $\eta(t)$ along the geodesic

$$s \mapsto \Gamma(s, t) := \exp_{z_\tau(t)}(sE(z_\tau(t), w(\tau, t))).$$

Now we write

$$\Pi^{-1} \nabla_t^\pi (\Pi \eta) - \nabla_{\dot{z}_\tau}^\pi \eta = \int_0^1 \frac{d}{ds} (\Pi_s^{-1} (\nabla_t^\pi (\Pi_s \eta))) ds \quad (10.7)$$

where Π_s is the parallel transport from z_τ to $\exp_{z_\tau}(sE(z_\tau, w(\tau, \cdot)))$. Then we compute

$$\begin{aligned} \frac{d}{ds} (\Pi_s^{-1} \nabla_t^\pi (\Pi_s \eta)) &= \Pi_s^{-1} \nabla_s^\pi (\nabla_t^\pi (\Pi_s \eta)) \\ &= \Pi_s^{-1} (\nabla_t^\pi \nabla_s^\pi (\Pi_s \eta)) + \Pi_s^{-1} K^\pi \left(\frac{\partial \Gamma}{\partial s}, \frac{\partial \Gamma}{\partial t} \right) (\Pi_s \eta) \\ &= \Pi_s^{-1} K^\pi \left(\frac{\partial \Gamma}{\partial s}, \frac{\partial \Gamma}{\partial t} \right) (\Pi_s \eta). \end{aligned}$$

But we note

$$\begin{aligned} \frac{\partial \Gamma}{\partial s} &= d_2 \exp_{z_\tau}(sE(z_\tau, w(\tau, \cdot)))(E(z_\tau, w(\tau, \cdot))) \\ \frac{\partial \Gamma}{\partial t} &= D_1 \exp_{z_\tau}(sE(z_\tau, w(\tau, \cdot))) \left(\frac{\partial z_\tau}{\partial t} \right) + d_2 \exp_{z_\tau}(sE(z_\tau, w(\tau, \cdot))) \left(s \frac{\partial E(z_\tau, w(\tau, \cdot))}{\partial t} \right). \end{aligned}$$

The constants C, C' appearing in the computations below may vary place but always depend only on the triad (Q, λ, J) and the $W^{1,2}$ -bound of w .

Using the equality $|E(x, y)| = d(x, y)$ when $d(x, y)$ is less than injective radius, it follows that

$$\begin{aligned} |d_2 \exp_{z_\tau}(sE(z_\tau, w(\tau, \cdot)))(E(z_\tau, w(\tau, \cdot)))| &\leq C |E(z_\tau, w(\tau, \cdot))| \leq C \|d(z_\tau, w(\tau, \cdot))\|_{C^0} \\ \left| D_1 \exp_{z_\tau}(sE(z_\tau, w(\tau, \cdot))) \left(\frac{\partial z_\tau}{\partial t} \right) \right| &\leq C \left| \frac{\partial z_\tau}{\partial t} \right| \leq CT \|X_\lambda\|_{C^0} = CT. \end{aligned}$$

On the other hand,

$$\begin{aligned} &\left| D_1 \exp_{z_\tau}(sE(z_\tau, w(\tau, \cdot))) \left(s \frac{\partial E(z_\tau, w(\tau, \cdot))}{\partial t} \right) \right| = \\ &\left| D_1 \exp_{z_\tau}(sE(z_\tau, w(\tau, \cdot))) s \left(D_1(E(z_\tau, w(\tau, \cdot))) \frac{\partial z_\tau}{\partial t} + d_2(E(z_\tau, w(\tau, \cdot))) \frac{\partial w}{\partial t} \right) \right|. \end{aligned}$$

It follows from the standard Jacobi field estimate [Ka]

$$\begin{aligned} \left| D_1(E(z_\tau, w(\tau, \cdot))) \frac{\partial z_\tau}{\partial t} + d_2(E(z_\tau, w(\tau, \cdot))) \frac{\partial w}{\partial t} \right| &\leq C \left(\left| \frac{\partial z_\tau}{\partial t} \right| + \left| \frac{\partial w}{\partial t} \right| \right) \\ &\leq C'T \end{aligned}$$

where the second inequality comes since $\frac{\partial z_\tau}{\partial t} = TX_\lambda(z_\tau)$ and $\left| \frac{\partial w}{\partial t} \right| \rightarrow T|X_\lambda| = T$ uniformly as $\tau \rightarrow \infty$. In summary, we have derived

$$\left| \frac{\partial \Gamma}{\partial s} \right| \leq C \|d(z_\tau, w(\tau, \cdot))\|_{C^0}, \quad \left| \frac{\partial \Gamma}{\partial t} \right| \leq C'T.$$

Therefore we have established

$$\left| \Pi_s^{-1} K^\pi \left(\frac{\partial \Gamma}{\partial s}, \frac{\partial \Gamma}{\partial t} \right) (\Pi_s \eta) \right| \leq CC'T \|d(z_\tau, w(\tau, \cdot))\|_{C^0} |\eta|.$$

We also recall $\|d(z_\tau, w(\tau, \cdot))\|_{C^0} \leq C \|d(z_\tau, w(\tau, \cdot))\|_{W^{1,2}}$. Substituting these into (10.7), there exists some τ_1 such that for all $\tau \geq \tau_1$, we obtain

$$|\Pi^{-1} \nabla_t^\pi(\Pi\eta) - \nabla_{z_\tau}^\pi \eta| \leq \frac{1}{8} |\eta|.$$

Substituting this into (10.6), we have obtained

$$\begin{aligned} |(\Pi^{-1} A_\tau \Pi \eta) - A_{z_\tau} \eta| &\leq |\Pi^{-1} \nabla_t^\pi(\Pi\eta) - \nabla_{z_\tau}^\pi \eta| + |\Pi^{-1} B \Pi(\eta) - DX_\lambda(z_\tau) \eta| \\ &\leq \frac{1}{8} |\eta| + \frac{1}{8} |\eta| = \frac{1}{4} |\eta|. \end{aligned}$$

Here we use the convergence $|\Pi^{-1} B \Pi(\eta) - DX_\lambda(z_\tau) \eta| = o(\tau) |\eta|$ from the expression of B (10.5) and the convergence

$$a_1 \rightarrow 0, \quad a_2 \rightarrow T$$

where the first convergence to zero follows from the assumption $a = 0$.

This finishes the proof of the lemma. \square

To prove

$$\|A_\tau(\eta)\|_{L^2} \geq \frac{3}{4} \lambda_1 \|\eta\|_{L^2} \quad (10.8)$$

it is enough to prove

$$|\langle A_\tau(\eta), \eta \rangle_{L^2}| \geq \frac{3}{4} \lambda_1 \langle \eta, \eta \rangle_{L^2}$$

for all $\eta \in W^{1,2}(w(\tau, \cdot)^* \xi)$. We write

$$\begin{aligned} |\langle A_\tau(\eta), \eta \rangle_{L^2}| &= |\langle \Pi^{-1} A_\tau \Pi \Pi^{-1}(\eta), \Pi^{-1} \eta \rangle_{L^2}| \\ &\geq |\langle A_{z_\tau}(\Pi^{-1} \eta), \Pi^{-1} \eta \rangle_{L^2}| - |\langle (\Pi^{-1} A_\tau \Pi - A_{z_\tau}(\Pi^{-1} \eta)), \Pi^{-1} \eta \rangle_{L^2}| \\ &\geq \lambda_1 \|\Pi^{-1} \eta\|_{L^2}^2 - \frac{1}{4} \lambda_1 \|\Pi^{-1} \eta\|_{L^2}^2 = \frac{3}{4} \lambda_1 \|\Pi^{-1} \eta\|_{L^2}^2 = \frac{3}{4} \lambda_1 \|\eta\|_{L^2}^2 \end{aligned}$$

where we use the fact that Π is an isometry. This now finishes the proof of the proposition. \square

Remark 10.5. The subtlety of the above proof of the lemma (and hence that of Proposition 9.5) lies in the fact that we would like to prove the uniform lower bound of the eigenvalues of the family A_τ for $\tau \geq \tau_2$ for all τ_2 . Because we only assume $w(\tau, \cdot)$ is $W^{1,2}$ -close to z_τ (even under the C^1 -closeness hypothesis), it is a nontrivial task for the a priori noncompact family of the operators A_τ whose domain and target, which are $W^{1,2}(w(\tau, \cdot)^* \xi)$ and $L^2(w(\tau, \cdot)^* \xi)$, have uniform bound below away from zero. Here enters the compactness of the embedding $W^{1,2} \hookrightarrow C^\epsilon$ and our careful usage of exponential map.

11. L^2 EXPONENTIAL CONVERGENCE OF ξ -COMPONENT OF dw

Under the hypothesis, Hypothesis 9.4, in addition to Hypothesis 6.2, we prove the following stronger convergence result in this section.

Proposition 11.1. *Under the Hypothesis 6.2 and 9.4, the T -periodic orbit given in Theorem 6.3 satisfies the following additional properties:*

$$\lim_{\tau \rightarrow +\infty} \left| \pi \frac{\partial w}{\partial \tau} \right| = 0 \quad (11.1)$$

$$\lim_{\tau \rightarrow +\infty} \left| \pi \frac{\partial w}{\partial t} \right| = 0, \quad (11.2)$$

$$\lim_{\tau \rightarrow +\infty} a_2(\tau, t) = \lim_{\tau \rightarrow +\infty} \lambda \left(\frac{\partial w}{\partial t} \right) = T \quad (11.3)$$

$$\lim_{\tau \rightarrow +\infty} a_1(\tau, t) = \lim_{\tau \rightarrow +\infty} \lambda \left(\frac{\partial w}{\partial \tau} \right) = a \quad (11.4)$$

uniformly in t , where a is determined by

$$a = - \int_{S^1} (w(0, \cdot)^* \lambda \circ j = \int_{S^1} \lambda \left(\frac{\partial w}{\partial \tau}(0, t) \right) dt.$$

Proof. By the closedness of $w^* \lambda \circ j$, it follows

$$\int_{S^1} w(\tau, \cdot)^* \lambda \circ j = \int_{S^1} w(0, \cdot)^* \lambda \circ j$$

for all $\tau \geq 0$. In addition, by recalling from (6.7), we have derived

$$w_\infty^* \lambda \left(\frac{\partial}{\partial \tau} \right) = a.$$

By the C^1 -convergence of $w(\tau_k, \cdot) \rightarrow w_\infty(0, \cdot)$, it follows

$$\int_{S^1} w_\infty(0, \cdot)^* \lambda \circ j - a \, d\tau = - \int_{S^1} a \, dt = -a$$

We also have

$$w_\infty^* \lambda \circ j = T \, d\tau - a \, dt, \quad w_\infty^* \lambda = T \, dt + a \, d\tau. \quad (11.5)$$

Next, we derive the derivative convergence. From the subsequence convergence proved above, it follows that for any sequence $\tau_k \rightarrow +\infty$, there exists a subsequence, still denoted by τ_k , such that $\lim_{k \rightarrow +\infty} w(\tau_k, \cdot) = z_\theta(\cdot)$ in C^1 topology on S^1 under Hypothesis (4) (The argument is exactly the same [HWZ1, HWZ2, Proposition 2.1], so we omit the details), where $z_\theta \in Z$ is some rotation of z whose associated rotation constant θ may depend on the choice of subsequence. Hence

$$w(\tau_k, \cdot) \rightarrow \dot{z}_\theta,$$

in $C^1(S^1)$, where $z_\theta \in Z$. Then we get

$$\lim_{k \rightarrow +\infty} \left| \pi \frac{\partial w}{\partial t}(\tau_k, t) \right| = 0, \quad (11.6)$$

$$\lim_{k \rightarrow +\infty} \lambda \left(\frac{\partial w}{\partial t} \right) = T. \quad (11.7)$$

Notice here, the period T does not depend on the choice of subsequence but determined by w .

Now if there exists some sequence such that $|\pi \frac{\partial w}{\partial t}|$ doesn't converge to zero, we can assume it has a subsequence converging to some non-zero constant. That is because we assume finite gradient bound in Hypothesis (2). Then, we can pick a subsequence again to make it converge to a Reeb orbit and the contradiction

appears. Similarly, we can do the same argument to $\lambda(\frac{\partial w}{\partial t})(\tau_k, t)$, and thus (11.3) follows.

From the Hypothesis (1), both

$$\lim_{\tau \rightarrow +\infty} \pi \frac{\partial w}{\partial \tau} = 0, \quad \lim_{\tau \rightarrow +\infty} a_2 = T$$

immediately follow. We only need to show

$$\lim_{\tau \rightarrow +\infty} a_1 = a. \quad (11.8)$$

Assume this fails to hold, i.e., there exists some $\epsilon > 0$ and a sequence (τ_k, t_k) with $\tau_k \rightarrow \infty$, such that

$$|a_1(\tau_k, \cdot) - a| > \epsilon.$$

Then we look at the translated sequence

$$w_k(\tau, t) = w(\tau + \tau_k, t)$$

again, and we get w_∞ in the same way. From (11.5), It follows that

$$0 = \left| w_\infty^* \lambda \left(\frac{\partial}{\partial \tau} \right) - a \right| = \lim_{k \rightarrow \infty} \left| w_k^* \lambda \left(\frac{\partial}{\partial \tau} \right) - a \right| = \lim_{k \rightarrow \infty} |a_1(\tau_k, \cdot) - a| \geq \epsilon,$$

which gives contradiction and we are done with the proof. \square

Now let (τ, t) be the coordinates of the given cylindrical metric near a given (positive) puncture of the Riemann surface (Σ, j) . From now on in the rest of the paper, we will restrict ourselves to the case of vanishing charge, i.e., we put the following hypothesis

Hypothesis 11.2 (Charge vanishing). We assume the asymptotic charges of w at all ends vanish, i.e.,

$$-a = \lim_{\tau \rightarrow \infty} \int_{\partial_\ell \Sigma(\rho)} w(\tau, 0)^* \lambda \circ j = 0 \quad (11.9)$$

for all $\ell = 1, \dots, k$ where $\rho = e^{-2\pi\tau}$.

We recall the definition of the function

$$f(\tau) = \frac{1}{2} \int_{S^1} e^\pi(\tau, t) dt, \quad e^\pi(\tau, t) = |d^\pi w|^2 = |\partial^\pi w|^2 = 2|\zeta(\tau, t)|^2.$$

Notice here, we have

$$f''(\tau) = \frac{d^2}{d\tau^2} \int_{S^1} |\zeta|^2 dt = -\frac{1}{2} \int_{S^1} \Delta e^\pi(\tau, t) dt.$$

We remark that in this formula Δe^π is the Hodge Laplacian

$$\Delta e^\pi = - \left(\frac{\partial^2 e^\pi}{\partial \tau^2} + \frac{\partial^2 e^\pi}{\partial t^2} \right),$$

the negative of the classical Laplacian. With this being said, the following is the main result in this section.

Theorem 11.3. *Assume Hypotheses 6.2, 9.4 and 11.2. Then there exist some constant $C > 0$, $\lambda_2 > 0$ and $\tau_0 > 0$ such that*

$$\int_{S^1} |\zeta(\tau, t)|^2 dt \leq C e^{-\lambda_2 \tau}$$

for all $\tau \geq \tau_0$. Here the constant C depends only on the triad (Q, λ, J) and the C^1 -norm of w provided in Hypothesis 6.2.

The rest of the section will be occupied by the proof of this theorem by estimating of the integral

$$\frac{1}{2} \int_{S^1} \Delta e^\pi dt.$$

For this purpose, we need to analyze the integrands Δe^π in Theorem 4.8 of the integral f and give explicit estimate of each term.

We recall from Theorem 4.8 that for any solution w

$$\begin{aligned} \frac{1}{2} \Delta e^\pi &= -|\nabla^\pi(\partial^\pi w)|^2 - 2\delta\langle *d^{\nabla^\pi} \partial^\pi w, * \partial^\pi w \rangle + 2|\delta^{\nabla^\pi} \partial^\pi w|^2 \\ &\quad - \langle K^\pi(dw, dw) \partial^\pi w, \partial^\pi w \rangle - R|\partial^\pi w|^2. \end{aligned} \quad (11.10)$$

Here the last term drops out on the punctured neighborhood where the flat cylindrical metric is equipped.

Next, we use Theorem 4.8 to show the exponential decay in cylinder coordinates. We recall the operator \bar{D} and Lemma 8.1:

$$\bar{D}\zeta = \nabla_\tau^\pi \zeta + J\nabla_t^\pi \zeta.$$

Then we have

$$\begin{aligned} \delta^{\nabla^\pi} \partial^\pi w &= -\bar{D}\zeta \\ *d^{\nabla^\pi} \partial^\pi w &= J\bar{D}\zeta. \end{aligned} \quad (11.11)$$

From the fundamental equation in the cylinder coordinate in Proposition 7.1, we have the following crucial ‘on-shell’ formula

$$\bar{D}\zeta = \frac{1}{2}a_2(\mathcal{L}_{X_\lambda} J)\zeta - \frac{1}{2}a_1(\mathcal{L}_{X_\lambda} J)J\zeta \quad (11.12)$$

From this expression, we obtain

Lemma 11.4.

$$|\bar{D}\zeta| \leq C|\zeta|$$

for some constant C which is independent of ζ but depends only on the geometry of (Q, λ, J) .

Under the cylindrical coordinates (τ, t) , we use the following formula for $|\nabla\alpha|$, where $\alpha = \zeta d\tau + \eta dt$ is a vector valued one-form. This is a vector-valued analog to the well-known Gårding’s equality

$$|\nabla\alpha|^2 = |d\alpha|^2 + |\delta\alpha|^2$$

for ordinary real valued one-forms. We extend the wedge product \wedge to the E -valued one-forms by taking the inner product in the fiber direction of $\Omega^1(E)$ and the wedge product on the base Σ for vector bundle $E \rightarrow \Sigma$.

Applying lemma 4.5 to our case $\nabla^\pi(\partial^\pi w)$ and using $\eta = J\zeta$ for a contact Cauchy-Riemann map, we compute the first two terms in Theorem 4.8 as follows.

$$\begin{aligned} &-|\nabla^\pi(\partial^\pi w)|^2 + 2|\delta^{\nabla^\pi} \partial^\pi w|^2 \\ &= -|d^{\nabla^\pi} \partial^\pi w|^2 - |\delta^{\nabla^\pi} \partial^\pi w|^2 + 4\langle \nabla_\tau^\pi \zeta, J\nabla_t^\pi \zeta \rangle + 2|\delta^{\nabla^\pi} \partial^\pi w|^2 \\ &= 4\langle \nabla_\tau^\pi \zeta, J\nabla_t^\pi \zeta \rangle \end{aligned} \quad (11.13)$$

$$\begin{aligned} &= 4\langle \nabla_\tau^\pi \zeta, \bar{D}\zeta - \nabla_\tau^\pi \zeta \rangle \\ &= -4|\nabla_\tau^\pi \zeta|^2 + 4\langle \nabla_\tau^\pi \zeta, \bar{D}\zeta \rangle. \end{aligned} \quad (11.14)$$

Now, we compute the third term in Theorem 4.8, i.e., $\delta \langle d^{\nabla^\pi} \partial^\pi w, \partial^\pi w \rangle$. We recall that for any vector valued one-form α ,

$$\delta^\nabla \alpha = - * d^\nabla * \alpha = - \frac{D}{\partial \tau} \left(\alpha \left(\frac{\partial}{\partial \tau} \right) \right) - \frac{D}{\partial t} \left(\alpha \left(\frac{\partial}{\partial \tau} \right) \right). \quad (11.15)$$

To apply this to d^{∇^π} , δ^{∇^π} and $\alpha = \langle *d^{\nabla^\pi} \partial^\pi w, * \partial^\pi w \rangle$, we compute

$$\begin{aligned} \alpha \left(\frac{\partial}{\partial \tau} \right) &= \left\langle *d^{\nabla^\pi} \partial^\pi w, * \partial^\pi w \right\rangle \left(\frac{\partial}{\partial \tau} \right) \\ &= \left\langle *d^{\nabla^\pi} \partial^\pi w, (* \partial^\pi w) \left(\frac{\partial}{\partial \tau} \right) \right\rangle = \langle *d^{\nabla^\pi} \partial^\pi w, -J\zeta \rangle, \end{aligned}$$

and

$$\begin{aligned} \alpha \left(\frac{\partial}{\partial t} \right) &= \langle *d^{\nabla^\pi} \partial^\pi w, * \partial^\pi w \rangle \left(\frac{\partial}{\partial t} \right) \\ &= \left\langle *d^{\nabla^\pi} \partial^\pi w, (* \partial^\pi w) \left(\frac{\partial}{\partial t} \right) \right\rangle = \langle *d^{\nabla^\pi} \partial^\pi w, \zeta \rangle. \end{aligned}$$

Using (11.15), we get

$$\delta \langle *d^{\nabla^\pi} \partial^\pi w, * \partial^\pi w \rangle = \frac{\partial}{\partial \tau} \langle *d^{\nabla^\pi} \partial^\pi w, J\zeta \rangle - \frac{\partial}{\partial t} \langle *d^{\nabla^\pi} \partial^\pi w, \zeta \rangle.$$

The second term will vanish after we take integral over S^1 for t parameter, so we only need to take care of the first term. By using metric property of ∇^π and (11.11), it can be written as

$$\frac{\partial}{\partial \tau} \langle *d^{\nabla^\pi} \partial^\pi w, J\zeta \rangle = \frac{\partial}{\partial \tau} \langle \bar{D}\zeta, \zeta \rangle = \langle \nabla_\tau^\pi (\bar{D}\zeta), \zeta \rangle + \langle \bar{D}\zeta, \nabla_\tau^\pi \zeta \rangle.$$

Hence together with (11.13), Theorem 4.8 can be written as the following expression under cylindrical coordinates.

$$\begin{aligned} \frac{1}{2} \Delta e^\pi &= -4 |\nabla_\tau^\pi \zeta|^2 + 2 (\langle \nabla_\tau^\pi \zeta, \bar{D}\zeta \rangle - \langle \nabla_\tau^\pi (\bar{D}\zeta), \zeta \rangle) \\ &\quad + 2 \frac{\partial}{\partial t} \langle *d^{\nabla^\pi} \partial^\pi w, \zeta \rangle - \langle K^\pi(dw, dw) \partial^\pi w, \partial^\pi w \rangle. \end{aligned} \quad (11.16)$$

We calculate the second term by looking at $\langle \nabla_\tau^\pi \zeta, \bar{D}\zeta \rangle - \langle \nabla_\tau^\pi (\bar{D}\zeta), \zeta \rangle$.

Using (11.12), we compute

$$\begin{aligned} \langle \nabla_\tau^\pi \zeta, \bar{D}\zeta \rangle &= \left\langle \nabla_\tau^\pi \zeta, -\frac{1}{2} a_1 (\mathcal{L}_{X_\lambda} J) J\zeta + \frac{1}{2} a_2 (\mathcal{L}_{X_\lambda} J) \zeta \right\rangle \\ &= -\frac{1}{2} a_1 \langle \nabla_\tau^\pi \zeta, (\mathcal{L}_{X_\lambda} J) J\zeta \rangle + \frac{1}{2} a_2 \langle \nabla_\tau^\pi \zeta, (\mathcal{L}_{X_\lambda} J) \zeta \rangle \end{aligned}$$

and

$$\begin{aligned} \langle \nabla_\tau^\pi (\bar{D}\zeta), \zeta \rangle &= \left\langle \nabla_\tau^\pi \left(\frac{1}{2} a_2 (\mathcal{L}_{X_\lambda} J) \zeta - \frac{1}{2} a_1 (\mathcal{L}_{X_\lambda} J) J\zeta \right), \zeta \right\rangle \\ &= -\frac{1}{2} \langle (\nabla_\tau^\pi (a_1 \mathcal{L}_{X_\lambda} J)) J\zeta, \zeta \rangle + \frac{1}{2} \langle (\nabla_\tau^\pi (a_2 \mathcal{L}_{X_\lambda} J)) \zeta, \zeta \rangle \\ &\quad - \frac{1}{2} a_1 \langle (\mathcal{L}_{X_\lambda} J) \nabla_\tau^\pi \zeta, J\zeta \rangle - \frac{1}{2} a_2 \langle (\mathcal{L}_{X_\lambda} J) \nabla_\tau^\pi \zeta, \zeta \rangle \end{aligned}$$

Subtracting these two terms and noting that $\mathcal{L}_{X_\lambda} J$ is symmetric, we get

$$\begin{aligned} & \langle \nabla_\tau^\pi \zeta, \bar{D}\zeta \rangle - \langle \nabla_\tau^\pi(\bar{D}\zeta), \zeta \rangle \\ &= \frac{1}{2} \langle (\nabla_\tau^\pi(a_1 \mathcal{L}_{X_\lambda} J)) J\zeta, \zeta \rangle - \frac{1}{2} \langle (\nabla_\tau^\pi(a_2 \mathcal{L}_{X_\lambda} J)) \zeta, \zeta \rangle. \end{aligned} \quad (11.17)$$

Now we estimate the two terms in (11.17).

For the first term in (11.17), we look at

$$\langle (\nabla_\tau^\pi(a_2 \mathcal{L}_{X_\lambda} J)) \zeta, \zeta \rangle = \frac{\partial a_2}{\partial \tau} \langle (\mathcal{L}_{X_\lambda} J) \zeta, \zeta \rangle + a_2 \langle (\nabla_\tau^\pi(\mathcal{L}_{X_\lambda} J)) \zeta, \zeta \rangle. \quad (11.18)$$

The second term of (11.18) is of $o(1)|\zeta|^2$ since the operator norm

$$\|\nabla_\tau^\pi(\mathcal{L}_{X_\lambda} J)\| \leq \|\nabla(\mathcal{L}_{X_\lambda} J)\|_{C^0} \left| \frac{\partial w}{\partial \tau} \right| = o(1), \quad (11.19)$$

where where we denote by $o(1)$ some function approaches zero in as $\tau \rightarrow \infty$ and $\|\nabla(\mathcal{L}_{X_\lambda} J)\|_{C^0}$ the gradient bound of the vector field $\mathcal{L}_{X_\lambda} J$ on Q which is independent of w but given by the geometry of (Q, λ, J) .

We note

$$\frac{\partial a_2}{\partial \tau} - \frac{\partial a_1}{\partial t} = *dw^* \lambda = \frac{1}{2} |\partial^\pi w|^2 = |\zeta|^2 = o(1). \quad (11.20)$$

The first term of (11.18),

$$\begin{aligned} & \frac{\partial a_2}{\partial \tau} \langle (\mathcal{L}_{X_\lambda} J) \zeta, \zeta \rangle \\ &= (*dw^* \lambda) \langle (\mathcal{L}_{X_\lambda} J) \zeta, \zeta \rangle + \frac{\partial a_1}{\partial t} \langle (\mathcal{L}_{X_\lambda} J) \zeta, \zeta \rangle \\ &= o(1)|\zeta|^2 + \frac{\partial}{\partial t} (a_1 \langle (\mathcal{L}_{X_\lambda} J) \zeta, \zeta \rangle) - a_1 \frac{\partial}{\partial t} \langle (\mathcal{L}_{X_\lambda} J) \zeta, \zeta \rangle \end{aligned} \quad (11.21)$$

Here for the last term (11.21), we have the following lemma.

Lemma 11.5.

$$\frac{\partial}{\partial t} \langle (\mathcal{L}_{X_\lambda} J) \zeta, \zeta \rangle = O(1)|\zeta|^2 + O(1)|\nabla_\tau^\pi \zeta|^2.$$

Here $O(1)$ denotes some bounded functions.

Proof. We compute

$$\begin{aligned} \frac{\partial}{\partial t} \langle (\mathcal{L}_{X_\lambda} J) \zeta, \zeta \rangle &= \langle \nabla_t^\pi((\mathcal{L}_{X_\lambda} J) \zeta), \zeta \rangle + \langle (\mathcal{L}_{X_\lambda} J) \zeta, \nabla_t^\pi \zeta \rangle \\ &= \langle (\nabla_t^\pi(\mathcal{L}_{X_\lambda} J)) \zeta, \zeta \rangle + \langle (\mathcal{L}_{X_\lambda} J) \nabla_t^\pi \zeta, \zeta \rangle + \langle (\mathcal{L}_{X_\lambda} J) \zeta, \nabla_t^\pi \zeta \rangle \\ &= \langle (\nabla_t^\pi(\mathcal{L}_{X_\lambda} J)) \zeta, \zeta \rangle + 2 \langle (\mathcal{L}_{X_\lambda} J) \nabla_t^\pi \zeta, \zeta \rangle \\ &= \langle (\nabla_t^\pi(\mathcal{L}_{X_\lambda} J)) \zeta, \zeta \rangle + 2 \langle J(\mathcal{L}_{X_\lambda} J) \zeta, \bar{D}\zeta \rangle - 2 \langle J(\mathcal{L}_{X_\lambda} J) \zeta, \nabla_\tau^\pi \zeta \rangle \\ &= \langle (\nabla_t^\pi(\mathcal{L}_{X_\lambda} J)) \zeta, \zeta \rangle + O(1)(|\zeta|^2 + |\nabla_\tau^\pi \zeta|^2). \end{aligned}$$

While the first term

$$\begin{aligned} & |\langle (\nabla_t^\pi(\mathcal{L}_{X_\lambda} J)) \zeta, \zeta \rangle| \\ &\leq \|\nabla(\mathcal{L}_{X_\lambda} J)\|_{C^0} \left| \frac{\partial w}{\partial t} \right| |\zeta|^2 = O(1)|\zeta|^2, \end{aligned}$$

and we are done with the proof. \square

Thus we have established that (11.21) is of the form

$$o(1)|\zeta|^2 + o(1)|\nabla_\tau^\pi \zeta|^2 + \frac{\partial}{\partial t}(a_1 \langle (\mathcal{L}_{X_\lambda} J)\zeta, \zeta \rangle).$$

For the second term of (11.17), we look at $-\langle (\nabla_\tau^\pi(a_1 \mathcal{L}_{X_\lambda} J))J\zeta, \zeta \rangle$. Similarly, we have

$$\begin{aligned} & \langle (-\nabla_\tau^\pi(a_1 \mathcal{L}_{X_\lambda} J))J\zeta, \zeta \rangle \\ &= -\frac{\partial a_1}{\partial \tau} \langle (\mathcal{L}_{X_\lambda} J)J\zeta, \zeta \rangle - a_1 \langle (\nabla_\tau^\pi(\mathcal{L}_{X_\lambda} J))J\zeta, \zeta \rangle \\ &= \frac{\partial a_2}{\partial t} \langle (\mathcal{L}_{X_\lambda} J)J\zeta, \zeta \rangle - a_1 \langle (\nabla_\tau^\pi(\mathcal{L}_{X_\lambda} J))J\zeta, \zeta \rangle \\ &= \frac{\partial a_2}{\partial t} \langle (\mathcal{L}_{X_\lambda} J)J\zeta, \zeta \rangle + o(1)|\zeta|^2 \end{aligned} \quad (11.22)$$

by the same reason as (11.19).

Remark 11.6. We would like to remark that this, or more specifically the second equality, is another place where we use the closedness of $w^* \lambda \circ j$, i.e., the ‘divergence free’ condition of $w^* \lambda$

$$\frac{\partial a_1}{\partial \tau} + \frac{\partial a_2}{\partial t} = 0$$

in an essential way to perform the integration by parts.

The first term (11.22) is dealt with similarly as (11.21).

$$\begin{aligned} & \frac{\partial a_2}{\partial t} \langle (\mathcal{L}_{X_\lambda} J)J\zeta, \zeta \rangle \\ &= \frac{\partial}{\partial t}(a_2 \langle (\mathcal{L}_{X_\lambda} J)J\zeta, \zeta \rangle) - a_2 \frac{\partial}{\partial t} \langle (\mathcal{L}_{X_\lambda} J)J\zeta, \zeta \rangle \\ &= \frac{\partial}{\partial t}(a_2 \langle (\mathcal{L}_{X_\lambda} J)J\zeta, \zeta \rangle) - T \frac{\partial}{\partial t} \langle (\mathcal{L}_{X_\lambda} J)J\zeta, \zeta \rangle - (a_2 - T) \frac{\partial}{\partial t} \langle (\mathcal{L}_{X_\lambda} J)J\zeta, \zeta \rangle \\ &= \frac{\partial}{\partial t}(a_2 \langle (\mathcal{L}_{X_\lambda} J)J\zeta, \zeta \rangle) - T \frac{\partial}{\partial t} \langle (\mathcal{L}_{X_\lambda} J)J\zeta, \zeta \rangle + o(1)|\zeta|^2 + o(1)|\nabla_\tau^\pi \zeta|^2. \end{aligned}$$

The reason for the last equality is exactly the same as the way we deal with (11.21) using Lemma 11.5, so we omit the details.

Above all, we now get the following estimate of energy density e^π under cylinder coordinates which is important to get L^2 exponential decay of ζ .

Theorem 11.7. *Let w be a contact instanton map. Then*

$$\begin{aligned} \frac{1}{2} \Delta e^\pi &= -(4 - o(1))|\nabla_\tau^\pi \zeta|^2 + o(1)|\zeta|^2 \\ &\quad + 2 \frac{\partial}{\partial t} \langle *d^{\nabla^\pi} \partial^\pi w, \zeta \rangle - \frac{\partial}{\partial t} (a_1 \langle (\mathcal{L}_{X_\lambda} J)\zeta, \zeta \rangle) \\ &\quad - \frac{\partial}{\partial t} (a_2 \langle (\mathcal{L}_{X_\lambda} J)J\zeta, \zeta \rangle) + T \frac{\partial}{\partial t} \langle (\mathcal{L}_{X_\lambda} J)J\zeta, \zeta \rangle. \end{aligned} \quad (11.23)$$

Notice the last two lines vanish after taking integral over S^1 for t .

Now let (τ, t) be the coordinates on the given cylindrical end and recall the function f is defined as

$$f(\tau) = \frac{1}{2} \int_{S^1} e^\pi(\tau, t) dt, \quad e^\pi(\tau, t) = |\partial^\pi w|^2 = 2|\zeta(\tau, t)|^2.$$

Then we have

$$f''(\tau) = \frac{1}{2} \int_{S^1} \frac{\partial^2 e^\pi}{\partial \tau^2}(\tau, t) dt = \frac{1}{2} \int_{S^1} -\Delta e^\pi(\tau, t) dt.$$

Then the above calculation together with Proposition 10.3 leads to the following proposition.

Proposition 11.8. *There exist some constant $\delta > 0$ and τ_0 large such that for any $\tau > \tau_0$,*

$$f(\tau)'' \geq \delta f(\tau)$$

Proof. We note

$$-\nabla_\tau \zeta = J\nabla_t + B(\zeta) = A_\tau(\zeta)$$

from the fundamental equation where B is the operator given in (10.5). On the other hand, integrating (11.23) over S^1 , we have

$$f''(\tau) = (4 - o(1)) \int_{S^1} |\nabla_\tau \zeta|^2 + o(1) \int_{S^1} |\zeta|^2$$

Applying the eigenvalue estimate in Proposition 10.3, we derive

$$f''(\tau) \geq 3 \int_{S^1} |\nabla_\tau \zeta|^2 + o(1) \int_{S^1} |\zeta|^2 \geq \left(3\left(\frac{2\lambda_1}{3}\right)^2 + o(1)\right) \int_{S^1} |\zeta|^2.$$

From here we immediately derive that there exists some τ_0 some $\delta > 0$ depending only on the first eigenvalue of the linearization operator A_z of the asymptotic orbit (or on the constant λ_1 given in Proposition 10.3 such that

$$f''(\tau) \geq \delta f(\tau)$$

for all $\tau \geq \tau'_0$. This finishes the proof. \square

A well-known standard maximum principle argument and the vanishing $f(\tau) \rightarrow 0$ as $\tau \rightarrow \infty$ concludes that $f(\tau)$ exponentially decays to zero. Hence we have finished the proof of Theorem 11.3.

12. ALTERNATING BOOTSTRAPPING AND C^∞ EXPONENTIAL CONVERGENCE OF dw

Recall the Hermitian connection ∇^π on the Hermitian bundle $\xi \rightarrow Q$ gives a Cauchy-Riemann operator \bar{D} defined by

$$\frac{1}{2}\bar{D} = \frac{\nabla^\pi + J\nabla^\pi(\cdot) \circ j}{2},$$

which we will apply ζ .

We have the elliptic estimate for the Cauchy-Riemann operator $\bar{\partial}^\pi$ on any closed regions $[l_0 + 1, l_1 - 1] \times S^1 \subset [l_0, l_1] \times S^1$

$$\begin{aligned} & \|\zeta\|_{W^{k,2}([l_0+1, l_1-1] \times S^1)} \\ & \leq c_{k, l_0, l_1} \left(\left\| \frac{1}{2}\bar{D}\zeta \right\|_{W^{k-1,2}([l_0, l_1] \times S^1)} + \|\zeta\|_{W^{k-1,2}([l_0, l_1] \times S^1)} \right) \end{aligned} \quad (12.1)$$

where c_{k, l_0, l_1} is some constant depending on k , l_0 and l_1 , and $k = 1, 2, \dots$.

We write the fundamental equation (7.4) into the form of

$$\frac{1}{2}\bar{D}\zeta + S\zeta = 0,$$

where

$$\frac{1}{2}\bar{D}\zeta = \frac{1}{2}(\nabla_\tau^\pi \zeta + J\nabla_t^\pi \zeta)$$

is a Cauchy-Riemann operator, and

$$S\zeta = \left(\frac{1}{2}a_1(\mathcal{L}_{X_\lambda} J)J - \frac{1}{2}a_2(\mathcal{L}_{X_\lambda} J) \right) \zeta.$$

Then the (12.1) gives

$$\begin{aligned} & \|\zeta\|_{W^{k,2}([l_0+1, l_1-1] \times S^1)} \\ & \leq c_{k, l_0, l_1} (\|S\zeta\|_{W^{k-1,2}([l_0, l_1] \times S^1)} + \|\zeta\|_{W^{k-1,2}([l_0, l_1] \times S^1)}). \end{aligned} \quad (12.2)$$

Now we proceed with several steps.

Step 1; $k = 1$ and for ζ : For $k = 1$, we estimate the right hand side on the region $[0, 7] \times S^1$ by

$$\|S\zeta\|_{L^2([0,7] \times S^1)} \leq \|S\|_{C^0} \|\zeta\|_{L^2([0,7] \times S^1)}. \quad (12.3)$$

where

$$\begin{aligned} \|S\|_{C^0} & \leq \frac{1}{4}(\|\mathcal{L}_{X_\lambda} J\|_{C^0} + \|J\|_{C^0} \|\mathcal{L}_{X_\lambda} J\|_{C^0}) \|\nabla w\|_{C^0} \\ & = \frac{1}{2}(\|\mathcal{L}_{X_\lambda} J\|_{C^0} \|\nabla w\|_{C^0} < \infty. \end{aligned}$$

We plug (12.3) into (12.2) and take $k = 1$. Then we get

$$\|\zeta\|_{W^{1,2}([1,6] \times S^1)} \leq c_{1,0,7}(1 + \|S\|_{C^0}) \|\zeta\|_{L^2([0,7] \times S^1)}.$$

Considering the translated sequence $\zeta \circ \tau$, we get

$$\|\zeta\|_{W^{1,2}([\tau+1, \tau+6] \times S^1)} \leq c_{1,0,7}(1 + \|S\|_{C^0}) \|\zeta\|_{L^2([\tau, \tau+7] \times S^1)},$$

for any $\tau \in \mathbb{R}$. Therefore, by considering $\tau \geq \tau_0$ in Theorem 11.3, we get

$$\|\zeta\|_{W^{1,2}([\tau+1, \tau+6] \times S^1)} \leq C' e^{-\delta' \tau},$$

where $\delta' = \frac{1}{2}\delta$ and C' is a constant given by the $W^{1,2}$ bound of ∇w and some other constants from the geometry of (Q, λ, J) . They are both independent of τ but only depend on $\|\nabla w\|_{C^0}$ and the contact triad (Q, λ, J) . To simplify the notation, we will always use C and δ for such kind of constants in this section. Then we get the following proposition.

Proposition 12.1. *There exist some constants $C > 0$ and $\tau_0 > 0$ such that for any $\tau > \tau_0$,*

$$\|\zeta\|_{W^{1,2}([\tau, +\infty) \times S^1)} \leq C e^{-\delta \tau}.$$

Step 2; for $k = 1$ and for a_1, a_2 : Next, we use Proposition 12.1 to get the L^2 exponential decay of X_λ part of dw by using the relation $*dw^* \lambda = |\zeta|^2$.

We define a complex-valued function

$$\theta(\tau, t) = (a_2 - T) - \sqrt{-1}a_1$$

and notice that θ satisfies the equation

$$\bar{\partial}\theta = \mu, \quad \mu = \frac{1}{2}(*dw^* \lambda) + \sqrt{-1} \cdot 0 = \frac{1}{2}|\zeta|^2 + \sqrt{-1} \cdot 0, \quad (12.4)$$

where $\bar{\partial} = \frac{1}{2} \left(\frac{\partial}{\partial \tau} - \sqrt{-1} \frac{\partial}{\partial t} \right)$ the standard Cauchy-Riemann operator for the standard complex structure $\sqrt{-1}$.

Notice that from Proposition 12.1 we have already established the $W^{1,2}$ exponential decay of μ . This gives rise to the L^2 exponential decay of θ as stated in the following proposition.

Proposition 12.2. *There exists some constants $C > 0$ and $\delta > 0$, such that*

$$\int_{S^1} |\theta|^2 dt < Ce^{-\delta\tau},$$

where $|\theta|^2 = (a_2 - T)^2 + a_1^2$.

The remaining section will be occupied by the proof of this proposition.

We start with the following general lemma which will be used several times for bootstrapping in this section. Proposition 12.2 immediately follows by considering $\theta = V$ and $\mu = W$. The proof is standard and is a much easier version of the argument used in the previous sections and so omitted.

Lemma 12.3. *Suppose that complex-valued functions θ and μ satisfy*

$$\bar{\partial}\theta = \mu$$

and

$$\begin{aligned} \|\mu\|_{L^2(S^1)} + \left\| \frac{\partial\mu}{\partial\tau} \right\|_{L^2(S^1)} &\leq Ce^{-\delta\tau}, \\ \lim_{\tau \rightarrow +\infty} \theta &= 0, \end{aligned}$$

then $\|\theta\|_{L^2(S^1)} \leq Ce^{-\delta\tau}$.

Remark 12.4. (1) We would like to remark here that the $L^2(S^1)$ exponential decay of a_1 actually can be obtained without assuming $W^{1,2}$ exponential decay of ξ , i.e., $L^2(S^1)$ exponential decay from Theorem 11.3 is enough. However, this $W^{1,2}$ exponential decay is crucial to get $L^2(S^1)$ exponential decay of $a_2 - T$.

(2) By applying the alternating bootstrapping arguments between $w^*\lambda$ and $d^\pi w$ for the higher derivatives used in section 8, we can inductively obtain the bound for $|\nabla^k dw|$ in terms of $|(\nabla^\pi)^l d^\pi w|$ and $|\nabla^l w^*\lambda|$ for $l \leq k-1$. Hence in section 13, we directly get the exponential decay of $|\nabla^k dw|$ once we get the exponential decay of $|(\nabla^\pi)^l d^\pi w|$ and $|\nabla^l w^*\lambda|$.

We now apply the standard elliptic bootstrapping and get

$$\|\theta\|_{W^{1,2}([1,6] \times S^1)} \leq c_{1,0,7} (\|\mu\|_{L^2([0,7] \times S^1)} + \|\theta\|_{L^2([0,7] \times S^1)}) \leq Ce^{-\delta\tau}.$$

Hence by using τ translation as above and we get $W^{1,2}$ exponential decay of θ .

Also, together with $W^{1,2}$ exponential decay of ζ and by using Remark 12.4 (2), we have now

$$\lim_{\tau \rightarrow +\infty} \|\nabla^2 w\|_{C^0} = 0 \tag{12.5}$$

$$\lim_{\tau \rightarrow +\infty} \|\nabla a_2\|_{C^0} = 0, \quad \lim_{\tau \rightarrow +\infty} \|\nabla a_1\|_{C^0} = 0 \tag{12.6}$$

Step 3; For $k = 2$ and for ζ : We go back to the elliptic estimate for ζ on the regions $[2, 5] \times S^1 \subset [1, 6] \times S^1$ for $k = 2$, and get

$$\|\zeta\|_{W^{2,2}([2,5] \times S^1)} \leq c_{2,1,6} \left(\|S\zeta\|_{W^{1,2}([1,6] \times S^1)} + \|\zeta\|_{W^{1,2}([1,6] \times S^1)} \right).$$

Notice $\|S\zeta\|_{W^{1,2}([1,6] \times S^1)}$ has the following estimate

$$\|S\zeta\|_{W^{1,2}([1,6] \times S^1)} \leq 3\|S\|_{C^1}\|\zeta\|_{W^{1,2}([1,6] \times S^1)},$$

where $\|S\|_{C^1}$ is bounded by the C^1 norm of ∇w and a_1, a_2 together with C^1 norm of ∇J and $\mathcal{L}_{X_\lambda} J$ on contact manifold Q . (12.5) and (12.6) guarantee it is bounded.

Hence similar elliptic bootstrapping argument as for $W^{1,2}$, we get for $W^{2,2}$ norm

$$\|\zeta\|_{W^{2,2}([\tau+2, \tau+5] \times S^1)} \leq Ce^{-\delta\tau},$$

and further

$$\|\zeta\|_{W^{2,2}([\tau, +\infty) \times S^1)} \leq Ce^{-\delta\tau}$$

when $\tau > \tau_0$, for some constants C and $\delta > 0$ which are independent of τ .

Step 4; for $k = 2$ and for ζ, a_1, a_2 : From the identity $*dw^*\lambda = |\zeta|^2$, Lemma 12.3 already implies that μ has $W^{2,2}$ exponential decay.

Now we differentiate equation (12.4), and get

$$\bar{\partial}\theta_\tau = \mu_\tau \tag{12.7}$$

Since μ_τ has $W^{1,2}$ exponential decay and from (12.6) we get

$$\lim_{\tau \rightarrow +\infty} \theta_\tau = 0,$$

we can apply Lemma 12.3 and get L^2 exponential decay of θ_τ . Then using elliptic bootstrapping to (12.7), we get $W^{1,2}$ exponential decay of θ_τ .

$W^{1,2}$ exponential decay of θ_t follows because

$$\theta_t = \sqrt{-1}(\theta_\tau - 2\mu)$$

from (12.4).

In summary, we now have $W^{2,2}$ exponential decay of both θ and ζ . By using Remark 12.4 (2) again, it indicates

$$\begin{aligned} \lim_{\tau \rightarrow +\infty} \|\nabla^3 w\|_{C^0} &= 0 \\ \lim_{\tau \rightarrow +\infty} \|\nabla^2 a_2\|_{C^0} &= 0, \quad \lim_{\tau \rightarrow +\infty} \|\nabla^2 a_1\|_{C^0} = 0. \end{aligned}$$

Step 5: Alternating elliptic bootstrap: Now the C^3 bound of ∇w and the C^2 bound of $\nabla^2 a_1$ and $\nabla^2 a_2$ guarantee us to do the above procedure again. For ζ , we will get

$$\|\zeta\|_{W^{3,2}([\tau+3, \tau+4] \times S^1)} \leq Ce^{-\delta\tau},$$

where C and $\delta > 0$ don't depend on τ .

By Sobolev embedding, $W^{3,2}([\tau+3, \tau+4] \times S^1)$ can be compactly embedded to $C^1([\tau+3, \tau+4] \times S^1)$, thus, we obtain the C^1 estimate

$$\|\zeta\|_{C^1([\tau+3, \tau+4] \times S^1)} \leq Ce^{-\delta\tau}.$$

Hence, further we get

$$\|\zeta\|_{C^1([\tau, +\infty) \times S^1)} \leq Ce^{-\delta\tau}$$

for $\tau > \tau_0$ when τ_0 is a large number.

Similarly, for θ , we get

$$\|\theta\|_{C^1([\tau, +\infty) \times S^1)} \leq Ce^{-\delta\tau}.$$

By induction of the above bootstrapping method, we get the following theorem for this section.

Theorem 12.5. *Under the same hypotheses as in Theorem 11.3, there exist some constants $C > 0$, $\delta > 0$ and τ_0 large such that for any $\tau > \tau_0$,*

$$\left\| \pi \frac{\partial w}{\partial \tau} \right\|_{C^\infty(S^1)} \leq Ce^{-\delta\tau} \quad (12.8)$$

$$\left\| \pi \frac{\partial w}{\partial t} \right\|_{C^\infty(S^1)} \leq Ce^{-\delta\tau} \quad (12.9)$$

$$\|a_2 - T\|_{C^\infty(S^1)} \leq Ce^{-\delta\tau} \quad (12.10)$$

$$\|a_1\|_{C^\infty(S^1)} \leq Ce^{-\delta\tau}. \quad (12.11)$$

13. C^0 EXPONENTIAL CONVERGENCE OF w (AND OF a)

By recalling the C^∞ exponential decay of dw from last section, we have the following weaker statement of the L^2 (over S^1) exponential decay of dw for any order $k \geq 0$ as $\tau \rightarrow \infty$.

Proposition 13.1. *There exist some constants $\delta > 0$ such that for each $k \geq 0$, whenever $\tau > \tau_0$,*

$$\int_{S^1} \left| \nabla^k \left(\frac{\partial w}{\partial \tau} \right) \right|^2 dt \leq Ce^{-\delta\tau} \quad (13.1)$$

$$\int_{S^1} \left| \nabla^k \left(\frac{\partial w}{\partial t} - TX_\lambda(w) \right) \right|^2 dt \leq Ce^{-\delta\tau}. \quad (13.2)$$

and for some universal constant $C = C_k > 0$ depending on k .

We would like to note that up to now, we have not obtained much about the C^0 asymptotic convergence of $w(\tau, \cdot)$ yet other than the subsequence asymptotic convergence in general given in Theorem 6.3 whose limit may depend on the choice of subsequence. Even in the current nondegenerate case, the rotation angle θ of the limit z_θ may depend on the choice of subsequences.

In this section, we finally proceed with the C^0 exponential convergence of $w(\tau, \cdot)$ to a Reeb orbit $z(\cdot)$ and also, under further assumption that $w^*\lambda \circ j$ is an exact form, i.e., there exists some function a such that $w^*\lambda \circ j = da$, $a(\tau, \cdot) \rightarrow T\tau + C$ for some constant C , provided we have subsequence convergence. For this, we will see that the L^2 exponential decay of dw itself, i.e., for $k = 0$ in Proposition 13.1, is enough to give the full C^0 convergence of $w(\tau, \cdot)$ as $\tau \rightarrow \infty$. To be more specific, (13.1) for $k = 0$ is enough to give the C^0 convergence of w to the nondegenerate Reeb orbit $z(\cdot)$ which is the main proposition in the first subsection. The inequality (13.2) for $k = 0$ is enough to give the C^0 convergence of a to a linear function which is the main proposition in the second subsection.

13.1. C^0 **exponential convergence of the map w .** The following is the main proposition we prove here.

Proposition 13.2. *Under Hypothesis 6.2, 9.4 and 11.2, there exists a unique Reeb orbit z with period $T > 0$ such that*

$$\|d(w(\tau, \cdot), z(\cdot))\|_{C^0(S^1)} \rightarrow 0,$$

as $\tau \rightarrow +\infty$.

Proof. We start with claiming that for each $t \in S^1$, $w(\cdot, t)$ is a Cauchy sequence.

If this claim is not true, then there exist some $t_0 \in S^1$ and some constant $\epsilon > 0$, sequences $\{\tau_k\}$, $\{p_k\}$ such that

$$d(w(\tau_{k+p_k}, t_0), w(\tau_k, t_0)) \geq \epsilon.$$

Then from the continuity of w in t , there exists some $l > 0$ small such that

$$d(w(\tau_{k+p_k}, t), w(\tau_k, t)) \geq \frac{\epsilon}{2}, \quad |t - t_0| \leq l.$$

Hence

$$\begin{aligned} & \int_{S^1} d(w(\tau_{k+p_k}, t), w(\tau_k, t)) \, dt \\ &= \int_{|t-t_0| \leq l} d(w(\tau_{k+p_k}, t), w(\tau_k, t)) \, dt + \int_{|t-t_0| > l} d(w(\tau_{k+p_k}, t), w(\tau_k, t)) \, dt \\ &\geq \int_{|t-t_0| \leq l} d(w(\tau_{k+p_k}, t), w(\tau_k, t)) \, dt \geq \epsilon l. \end{aligned}$$

On the other hand, we compute

$$\begin{aligned} & \int_{S^1} d(w(\tau_{k+p_k}, t), w(\tau_k, t)) \, dt \\ &\leq \int_{S^1} \int_{\tau_k}^{\tau_{k+p_k}} \left| \frac{\partial w}{\partial s}(s, t) \right| \, ds \, dt \\ &= \int_{\tau_k}^{\tau_{k+p_k}} \int_{S^1} \left| \frac{\partial w}{\partial s}(s, t) \right| \, dt \, ds \\ &\leq \int_{\tau_k}^{\tau_{k+p_k}} \left(\int_{S^1} \left| \frac{\partial w}{\partial s}(s, t) \right|^2 \, dt \right)^{\frac{1}{2}} \, ds \\ &\leq \int_{\tau_k}^{\tau_{k+p_k}} C e^{-\delta s} \, ds \\ &= \frac{C}{\delta} (1 - e^{-(\tau_{k+p_k} - \tau_k)}) e^{-\tau_k} \leq \frac{C}{\delta} e^{-\tau_k}. \end{aligned}$$

Hence we can take τ_k large and get contradiction.

Now by using the subsequence convergence from Theorem 6.3, we can pick an arbitrary subsequence $\{\tau_k\}$ and $z \in Z$ such that

$$w(\tau_k, t) \rightarrow z(t), \quad k \rightarrow \infty$$

uniformly in t . Then immediately from the fact that $w(\cdot, t)$ is a Cauchy sequence for any t , we get for any $t \in S^1$,

$$w(\tau, t) \rightarrow z(t), \quad \tau \rightarrow \infty.$$

What left to show is just this convergence is uniform in t , i.e., it is in $C^0(S^1)$ sense. Assume this is not true. Then there exist some $\epsilon > 0$ and some sequence (τ_k, t_k) such that

$$d(w(\tau_k, t_k), z(t_k)) \geq 2\epsilon.$$

Since $t_k \in S^1$, we can further take subsequence, still denote by t_k , such that $t_k \rightarrow t_0 \in S^1$. We can take k large such that $d(z(t_k), z(t_0)) \leq \frac{1}{2}\epsilon$.

We also look at

$$d(w(\tau, t_k), w(\tau, t_0)) \leq \int_{t_0}^{t_k} \left| \frac{\partial w}{\partial t}(\tau, s) \right| ds \leq (t_k - t_0) \|\nabla w\|_{C^0},$$

and so we can make it less than $\frac{1}{2}\epsilon$ by taking k large.

On the other hand, we have

$$\begin{aligned} d(w(\tau_k, t_0), z(t_0)) &\geq d(w(\tau_k, t_k), z(t_k)) - d(w(\tau_k, t_k), w(\tau_k, t_0)) \\ &\quad - d(z(t_k), z(t_0)) \\ &\geq 2\epsilon - \frac{1}{2}\epsilon - \frac{1}{2}\epsilon = \epsilon, \end{aligned}$$

which gives contradiction to the pointwise convergence.

This finishes the proof. \square

13.2. C^0 exponential convergence of a in the symplectization case. Finally we relate our general study of contact Cauchy-Riemann map to the special exact case, i.e. the case of maps (a, w) into the symplectization $\mathbb{R} \times Q$. In this case $\tilde{w} \equiv w$. (Here we follow the notation used by Hofer in [H1] to denote the \mathbb{R} component by a , although we have used a to denote the contact charge in the previous sections, which should not confuse the readers.)

In other words, we prove the C^0 convergence of a , assuming that

$$w^* \lambda \circ j = da.$$

In this case, we have

$$a_2 = \frac{\partial a}{\partial \tau}, \quad a_1 = -\frac{\partial a}{\partial t}$$

and the pair (a, w) satisfies the standard pseudoholomorphic curve equation

$$\bar{\partial}^\pi w = 0, \quad w^* \lambda \circ j = da.$$

Proposition 13.3. *There exists some constant C_0 , such that*

$$\|a(\tau, \cdot) - T\tau - C_0\|_{C^0(S^1)} \rightarrow 0,$$

as $\tau \rightarrow +\infty$.

Proof. Define $b(\tau, t) = a(\tau, t) - T\tau$. Then we have

$$\begin{aligned} \frac{\partial b}{\partial \tau} &= a_2 - T \rightarrow 0, \\ \frac{\partial b}{\partial t} &= -a_1 \rightarrow 0, \end{aligned}$$

as $\tau \rightarrow +\infty$ in $C^0(S^1)$ topology.

Define $\alpha(\tau) = \int_{S^1} b(\tau, t) dt$ and $\tilde{b}(\tau) = b(\tau, t) - \alpha(\tau)$. Then

$$|\alpha'(\tau)| = \left| \int_{S^1} \frac{\partial b}{\partial \tau} dt \right| \leq \int_{S^1} |a_2 - T| dt \leq C e^{-\delta \tau},$$

which indicates that $\alpha(\tau)$ is a Cauchy sequence. Then there exists some constant C_0 such that $\alpha(\tau) \rightarrow C_0$ as $\tau \rightarrow +\infty$.

On the other hand, notice here $\int_{S^1} \tilde{b}(\tau, t) dt = 0$ and then for any τ , there exists some point $t_0 \in S^1$ such that $\tilde{b}(\tau, t_0) = 0$. Then for any τ ,

$$|\tilde{b}(\tau, t)| = |\tilde{b}(\tau, t) - \tilde{b}(\tau, t_0)| \leq |t - t_0| \left\| \frac{\partial \tilde{b}}{\partial t} \right\|_{C^0(S^1)} \leq \left\| \frac{\partial \tilde{b}}{\partial t} \right\|_{C^0(S^1)} = \left\| \frac{\partial b}{\partial t} \right\|_{C^0(S^1)},$$

and thus

$$\|\tilde{b}(\tau, \cdot)\|_{C^0(S^1)} \leq \left\| \frac{\partial b}{\partial t} \right\|_{C^0(S^1)} = \|a_1\|_{C^0(S^1)} \rightarrow 0$$

as $\tau \rightarrow +\infty$. Hence

$$\|a(\tau, \cdot) - T\tau - C_0\|_{C^0(S^1)} \leq \|\tilde{b}(\tau, \cdot)\|_{C^0(S^1)} + |\alpha(\tau) - C_0| \rightarrow 0,$$

as $\tau \rightarrow +\infty$. We are done with the proof. \square

Now the C^0 exponential decay immediately follows from Theorem 12.5 and the C^0 convergence, Proposition 13.2 and Proposition 13.3.

Theorem 13.4. *There exist some constants $C > 0$, $\delta > 0$ and τ_0 large such that for any $\tau > \tau_0$,*

$$\begin{aligned} \|d(w(\tau, \cdot), z(\cdot))\|_{C^0(S^1)} &\leq C e^{-\delta\tau} \\ \|a(\tau, \cdot) - T\tau - C_0\|_{C^0(S^1)} &\leq C e^{-\delta\tau} \end{aligned}$$

Proof. For any $\tau < \tau_+$, similarly as in previous proof,

$$d(w(\tau, t), w(\tau_+, t)) \leq \int_{\tau}^{\tau_+} \left| \frac{\partial w}{\partial \tau}(s, t) \right| ds \leq \frac{C}{\delta} e^{-\delta\tau}.$$

Take $\tau_+ \rightarrow +\infty$ and using the C^0 convergence of w part, i.e., Proposition 13.2, we get

$$d(w(\tau, t), z(t)) \leq \frac{C}{\delta} e^{-\delta\tau}.$$

This proves the first inequality.

Similarly, we have

$$|(a(\tau_+, t) - T\tau_+ - C_0) - (a(\tau, t) - T\tau - C_0)| \leq \int_{\tau}^{\tau_+} |a_2(s, t) - T| ds \leq \frac{C}{\delta} e^{-\delta\tau},$$

where the last inequality comes from the C^0 exponential decay of $|a_2(s, \cdot) - T|$ in 13.3. By taking $\tau_+ \rightarrow +\infty$, we are done with the proof of the second inequality. \square

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